of our map is a maximal ideal and we see that Statements S2 and S3 are violated. This concludes the proof of our theorem.

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ON ORDERED SKEW FIELDS

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In this paper we shall give a necessary and sufficient condition that a skew field can be ordered; moreover, that the ordering of an ordered skew field K can be extended to an ordering of L, L being a given extension of K. The first of these two results generalizes to skew fields a theorem of E. Artin and O. Schreier [1],[1] according to which a commutative field can be ordered if and only if it is formally real. The second result generalizes in the same sense a recent theorem of J. P. Serre [2].

Our considerations are based on the following definition.

Definition. A skew field is said to be ordered if in its multiplicative group a subgroup of index 2 is marked out which is also closed under addition.

Hence a skew field can be ordered if and only if its multiplicative group has a subgroup of index 2 which is also closed under addition.

We shall now prove the following theorem.

Theorem 1. *A skew field K can be ordered if and only if -1 cannot be represented as a sum of elements of the form*

$$a_1^2 a_2^2 \cdots a_k^2 \quad (a_i \in K,\ i = 1, 2, \cdots, k). \tag{1}$$

Remark. This property can be considered as a generalization of the notion "formally real" to the case of skew fields.

The necessity of the condition in Theorem 1 is obvious. In order to prove its sufficiency we consider a skew field K in which -1 cannot be represented as a sum of elements (1). We shall show that the

Received by the editors September 20, 1951.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

multiplicative group K^* of K has a subgroup of index 2 which is also closed under addition.

Let S be the subset of all (finite) sums of elements (1) in K with every $a_i \neq 0$. Clearly $0 \notin S$, for in the contrary case we should have a relation

$$-a_1^2 a_2^2 \cdots a_k^2 = b_1^2 b_2^2 \cdots b_l^2 + \cdots$$

from which would follow, by multiplication on the right by $a_k^{-2}, \cdots, a_1^{-2}$, that $-1 \in S$ in contradiction to our hypothesis. On the other hand, one can see immediately that

$$s \in S,\ s' \in S \text{ imply } ss' \in S \text{ and } s + s' \in S,$$

$$s \in S \text{ implies } s^{-1} = s \cdot s^{-2} \in S,$$

$$s \in S,\ z \in K^* \text{ imply } z^{-1}sz \in S.$$

Hence S is a proper invariant subgroup of K^* which is closed under addition. The order of each element $(\neq 1)$ in K^*/S being 2, K^*/S is abelian. Consequently *any subgroup P of K^* which contains S is invariant in K^*.*

Now we define P as a *maximal* subgroup of K^* for which

$$S \subseteq P, \quad -1 \notin P, \text{ and } P \text{ is closed under addition.} \tag{2}$$

The existence of such a group P follows immediately from Zorn's lemma. We have only to show that the decomposition

$$K^* = P \cup (-1)P \tag{3}$$

holds. Suppose (3) is not true. Then there exists an element d such that

$$d \in K^*, \qquad d \notin P, \qquad -d \notin P. \tag{4}$$

Consider the set P' of all elements

$$u + vd \qquad (u, v \in \{P, 0\} \text{ but not } u = v = 0).$$

Then, by (4), P' contains P as a proper subset. On the other hand we shall show that P' is a subgroup of K^* having the properties (2) (with P' instead of P). This is a contradiction to the maximal property of P, which will complete the proof.

First we show that $0 \notin P'$. Indeed, by the exclusion of $u = v = 0$, $u + vd = 0$ would imply that $v \neq 0$ and hence that $-d = v^{-1}u \in P$, in contradiction to (4). Moreover, if $u_1 + v_1 d$ and $u_2 + v_2 d$ are arbitrary elements of P', we have

(5) $$(u_1 + v_1d)(u_2 + v_2d) = (u_1u_2 + v_1dv_2d) + (u_1v_2d + v_1du_2).$$

But since P is an invariant subgroup of K^*, $dv_2 = v_2'd$, $du_2 = u_2'd$ hold with suitable elements u_2', $v_2' \in P$, so that (5) is an element of P'. If $u+vd \in P'$, we obtain

$$(u + vd)^{-1} = (u + vd)(u + vd)^{-2} \in P'.$$

Hence P' is a group which is obviously closed under addition. Finally, $-1 \notin P'$ for $u+vd = -1$ would imply (on account of $v \neq 0$) that $-d = v^{-1}(u+1) \in P$. This completes the proof.

In an analogous manner we prove the following theorem.

THEOREM 2. *Let L be an extension of the ordered skew field K. The ordering of K can be extended to an ordering of L if and only if -1 cannot be represented as a sum of elements*

(6) $$p_1u_1^2 \cdots p_ku_k^2 \qquad (p_i \in K,\ p_i > 0,\ u_i \in L,\ i = 1, 2, \cdots, k).$$

REMARK. Theorem 1 is the special case of Theorem 2 in which K is the prime field of characteristic zero. However, this special case seemed of sufficient interest to warrant an independent proof. Only a few remarks are now necessary to prove Theorem 2 since the proof follows the same general pattern as that of Theorem 1.

The necessity of the condition in Theorem 2 is obvious. In order to prove its sufficiency we define the subset U of L as the set of all (finite) sums of elements (6) with every $u_i \neq 0$. One can show as above that U is a subgroup of the multiplicative group L^* of L. That, e.g., $0 \notin U$ follows from the fact that a relation

$$-p_1u_1^2 \cdots p_ku_k^2 = p_1'v_1^2 \cdots p_l'v_l^2 + \cdots$$

would imply that

$$-1 = p_1'v_1^2 \cdots p_l'v_l^2(1\cdot u_k^{-2})(p_k^{-1}u_{k-1}^{-2}) \cdots (p_1^{-1}\cdot 1^2) + \cdots,$$

which is impossible.

Moreover U is an invariant subgroup of L^*. This follows from the fact that

$$p \in K, \qquad p > 0, \qquad z \in L^*$$

imply that

$$z^{-1}pz = z^{-2}zpzpp^{-1} = p_1'v_1^2p_2'v_2^2p_3'$$

with $p_1' = 1$, $v_1 = z^{-1}$, $p_2' = 1$, $v_2 = zp$, $p_3' = p^{-1}$.

From the fact that each element ($\neq 1$) of L^*/U is of order 2, we infer as above that any subgroup Q of L^* containing U is invariant in L^*.

Now we define Q as a maximal subgroup of L^* for which $U \subseteq Q$, $-1 \notin Q$, and Q is closed under addition. Then one can show as above that Q is a subgroup of index 2 of L^*. Since all positive elements of K are contained in U and consequently in Q, the theorem is proved.

REMARK (added April 28, 1952). I have noticed that for the case of a countable skew field K, Theorem 1 occurs in the book of G. Pickert, *Einführung in die höhere Algebra* (Göttingen, 1951), p. 238, Aufgabe 13.

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