ON ORDERED DOMAINS OF INTEGRITY

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In a recent paper,[1] T. Szele proved that a division ring D is orderable if and only if the additive and multiplicative semigroup S generated by the nonzero squares of elements of D does not contain the zero element of D. The present paper extends this result to a domain of integrity K.

Let us denote by K^* the set of nonzero elements of K. The domain of integrity K is said to be orderable if and only if there exists an additive and multiplicative semigroup P (the positive elements) contained in K^* such that $K^* = P \cup (-P)$.

If K does not have a unit element, then there exists a unique minimal domain of integrity $\bar{K}$ having a unit element and containing K.[2] It is not too difficult to show that K is orderable if and only if $\bar{K}$ is orderable. For this reason, we assume henceforth that K has a unit element.

An element a of K^* is called *even* if there exist elements $a_1, \cdots, a_n$ in K^* such that a is a product of the $2n$ elements $a_1, \cdots, a_n$, $a_1, \cdots, a_n$ in some order. We denote by S the additive semigroup generated by the even elements of K^*. The theorem we wish to prove is as follows.

THEOREM. *The domain of integrity K is orderable if and only if $S \subset K^*$.*

The proof of this theorem will come after a few preliminary remarks. For any subset A of K, an element b of K is called *even* over A if there exist elements $a_1, \cdots, a_n$ of A and $k_1, \cdots, k_m$ of K^* such that b is a product of the $2m+n$ elements

$$a_1, \cdots, a_n, \; k_1, \cdots, k_m, \; k_1, \cdots, k_m$$

in some order. We denote by A^E the set of all elements of K even over A. Evidently A^E is a multiplicative semigroup containing A, and $A^{EE} = A^E$. Furthermore, if B is the additive semigroup generated by A^E, then $B^E = B$. Let $\mathfrak{S}$ denote the set of all subsets A of K such that (i) A is an additive semigroup, and (ii) $A^E = A$. Finally,

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[1] T. Szele, *On ordered skew fields*, Proceedings of the American Mathematical Society vol. 3 (1952) pp. 410–413.

[2] J. Szendrei, *On the extension of rings without divisors of zero*, Acta. Univ. Szeged. vol. 13 (1950) pp. 231–234.

for any subset A of K, denote by A^I the set of all elements b of K for which there exists an element a in A such that ba is in A.

LEMMA 1. *If A is in $\mathfrak{S}$, then A^I is in $\mathfrak{S}$ also.*

PROOF. If $b \in A^{IE}$, then b is a product of elements

$$a_1, \cdots, a_n, k_1, \cdots, k_m, k_1, \cdots, k_m, \qquad a_i \in A^I, k_i \in K^*,$$

in some order. Since each $a_i \in A^I$, there exist $c_i \in A$ such that $a_i c_i \in A$. Let $a = (a_1c_1) \cdots (a_nc_n)$, an element of A. If some $a_i = 0$, then $b = 0$ and $b \in A^I$ immediately. Otherwise, if all $a_i \neq 0$, $ba \in A$ since ba is a product of the elements

$$c_1, \cdots, c_n, a_1, \cdots, a_n, a_1, \cdots, a_n, k_1, \cdots, k_m, k_1, \cdots, k_m,$$

$c_i \in A$, a_i, $k_i \in K^*$, in some order. Thus $b \in A^I$ and $A^{IE} = A^I$.

To prove that A^I is an additive semigroup, let $a_i \in A^I$ with $a_i \neq 0$. Then there exist $c_i \in A$ such that $a_i c_i \in A$. Since

$$(a_1 + a_2)c_1a_2c_2 = (a_1c_1)(a_2c_2) + a_2c_1a_2c_2,$$

and $c_1(a_2c_2)$, $(a_1c_1)(a_2c_2)$, and $a_2c_1a_2c_2$ all are in A, we conclude that $a_1 + a_2 \in A^I$. Thus $A^I \in \mathfrak{S}$, and the lemma is proved.

LEMMA 2. *If $A \in \mathfrak{S}$, $A \subset K^*$, and $A^I = A$, and if, for $d \in K^*$, neither d nor $-d$ is in A, then the element B of $\mathfrak{S}$ generated by $A \cup (d)$ also is contained in K^*.*

PROOF. If $C = [A \cup (d)]^E$, then B is the additive semigroup generated by C. If $c \in C$, then either $c \in A$ or c is a product of elements

$$d, a_1, \cdots, a_n, k_1, \cdots, k_m, k_1, \cdots, k_m, \qquad a_i \in A, k_i \in K^*,$$

in some order. In this latter case, $dc \in A$. Thus $C = A \cup F$, where $dF \subset A$; and $B = A \cup F' \cup (A + F')$, where F' is the additive semigroup generated by F. Since $dF' \subset A$, evidently $F' \subset K^*$. To prove that $B \subset K^*$, let us assume $0 \in B$. Then $0 = a + f$ for some $a \in A$ and $f \in F'$. Since $(-d)(-f) \in A$, also $(-d)a \in A$. However $A^I = A$, and therefore $-d \in A$. This contradiction proves that $B \subset K^*$, and the lemma follows.

PROOF OF THEOREM. If K is orderable, so that $K^* = P \cup (-P)$ for some additive and multiplicative semigroup P, then it is easy to see that $S \subseteq P \subset K^*$.

On the other hand, if $S \subset K^*$, let $\mathfrak{A}$ be the subset of $\mathfrak{S}$ containing all A such that $S \subseteq A \subset K^*$. By Zorn's lemma, $\mathfrak{A}$ has a maximal element P. In view of Lemma 1, we must have $P^I = P$. That $K^* = P \cup (-P)$ is now an immediate consequence of Lemma 2.

In case K is a division ring, the set S coincides with the set generated by the perfect squares as used by Szele. This follows easily from the identity $xyx=(xy)^2(y^{-1})^2y$, x, $y\in K^*$.

If the domain of integrity K is ordered, say $K^*=P\cup(-P)$, then for an extension L of K the ordering of K can be extended to an ordering of L if and only if $T\subset L^*$, where T is the additive semigroup in L generated by P^E. The proof of this result is much the same as that of the above theorem. This generalizes Theorem 2 of Szele's paper to a domain of integrity.

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THE ZEROS OF AN ANALYTIC FUNCTION OF ARBITRARILY RAPID GROWTH[1]

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1. **Introduction.** It was shown by Poincaré [4],[2] Borel [1], and others that an integral function may be made "to grow" arbitrarily rapidly along the real axis or along other curves extending to infinity. Ketchum [2] has considered the corresponding problem for more general point sets. He investigated sets such that, for any given function $G(z)\geqq 0$, there exists a function $f(z)$ which is analytic except where $G(z)$ is unbounded and which satisfies the inequality

$$|f(z)|\geqq G(z)$$

for every point z of the set.

In the publication of his results Ketchum [2] proposed a corresponding problem in which the additional restriction is placed on the function $f(z)$ that it be nonvanishing except at certain specified points of the complement of the set. In particular, suppose $S_1, S_2, \cdots$ is an infinite sequence of simply-connected regions whose closures are nonintersecting and whose only "sequential limit point" is the point at infinity. Then, if $\{M_i\}$ is any preassigned sequence of positive constants, does there exist a nonvanishing integral function $f(z)$ such that $|f(z)|\geqq M_i$ when $z\in S_i$?

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[2] Numbers in brackets refer to the bibliography at the end of the paper.