A PROBABILISTIC APPROACH TO A SYSTEM OF INTEGRAL EQUATIONS[1]

LEO A. GOODMAN

The following system of integral equations is of some statistical interest:

$$G_n(y) = \int_0^y \frac{e^{-\omega}\omega^{n\alpha-1}}{\Gamma(n\alpha)} d\omega = \int_0^{g(y)} H_{n-1}(g(y) - z; g) dH_1(z; g) \tag{1}$$

$$\text{for } n = 1, 2, \cdots,$$

where $g(x) \geqq 0$ is an increasing continuous function of $x \geqq 0$, $H_0(x; g) = 1$, and

$$H_n(x; g) = \int_0^x H_{n-1}(x - z; g) dH_1(z; g).$$

We shall show by a probabilistic approach (distribution functions, the moment problem, etc.) that the only functions satisfying this system of equations for fixed α, $0 < \alpha \leqq 2$, are $g(x) = cx$, where c is a constant. It will then be seen that this result is a probabilistic analogue of the well known Cauchy functional equation. Also an application of this result to statistics is presented.

THEOREM 1. *If $g(x)$ satisfies the Cauchy functional equation*

$$g(x + y) = g(x) + g(y), \tag{2}$$

then $g(x)$ satisfies (1).

PROOF. For fixed n, we may define a random variable Y by the relation $\Pr\{Y \leqq y\} = G_n(y)$. Since $H_1(v; g) = \int_0^{f(v)} (e^{-\omega}\omega^{\alpha-1}/\Gamma(\alpha)) d\omega$, where $f(v)$ is the inverse function of $g(x)$, we may define a random variable V by the relation

$$\Pr\{V \leqq v\} = H_1(v; g).$$

Hence $H_n(v; g)$ may be regarded as the distribution of the sum of n independent random variables V_i each distributed as V. Equation (1) then states

$$\begin{aligned} \Pr\{Y \leqq y\} &= \Pr\{V_1 + V_2 + \cdots + V_n \leqq g(y)\} \\ &= \Pr\{f(V_1 + V_2 + \cdots + V_n) \leqq y\}. \end{aligned} \tag{3}$$

Received by the editors April 26, 1951.

[1] This research was prepared under an Office of Naval Research contract.

When $n=1$, we may write $Y=f(V_1)$. By the addition theorem for gamma distributions, we have

$$\begin{aligned}(4)\qquad \Pr\{f(V_1)+f(V_2)+\cdots+f(V_n)\leqq y\} &= G_n(y)\\ &= \Pr\{f(V_1+V_2+\cdots+V_n)\leqq y\}.\end{aligned}$$

Since when g satisfies (2), so does f, a fortiori (4) and (1) are satisfied by f and g. Q.E.D.

THEOREM 2. *The only functions satisfying* (1) *for fixed* α, $0<\alpha\leqq 2$, *are* $g(x)=cx$, *where* c *is a constant.*

PROOF. We use the notation developed in the proof of Theorem 1. Then $g(Y)$ is a random variable and by (3) has the same distribution function as the sum of n independent random variables V_i. Let $\phi(t) =\int_0^\infty e^{-tx}dG_1(f(x))$, for fixed $t>0$. Then by the fundamental property of moment generating functions, $\int_0^\infty e^{-tx}dG_n(f(x))=\phi^n(t)$. Hence, we have the system of equations

$$(5)\qquad \int_0^\infty e^{-tg(\omega)-\omega}\frac{\omega^{n\alpha-1}}{\Gamma(n\alpha)}\,d\omega = \phi^n(t).$$

Let us define a new random variable Z which has the distribution function

$$(6)\qquad \Pr\{Z\leqq z\} = \int_0^z \frac{e^{-tg(\omega)-\omega}}{\phi\Gamma(\alpha)}\,\omega^{\alpha-1}d\omega, \qquad \text{where } \phi=\phi(t).$$

Hence by (5) we have that the $n-1$ moment of Z^α/ϕ is equal to $\Gamma(n\alpha)/\Gamma(\alpha)$. It can easily be seen that $1, \Gamma(2\alpha)/\Gamma(\alpha), \Gamma(3\alpha)/\Gamma(\alpha), \cdots$ is the sequence of moments of the density function

$$\frac{e^{-h^{1/\alpha}}}{\Gamma(\alpha+1)}\,dh.$$

By (6), the density function of $Z^\alpha/\phi=k$ is

$$\frac{e^{-tg((\phi k)^{1/\alpha})-(\phi k)^{1/\alpha}}}{\Gamma(\alpha+1)}\,dk.$$

Since the moment problem is determined for $\alpha\leqq 2$ (cf. [1]), we have that

$$k^{1/\alpha}\left[\frac{1-\phi^{1/\alpha}}{t}\right] = g((\phi k)^{1/\alpha})$$

or

$$g(x) = \left[\frac{1 - \phi^{1/\alpha}}{t\phi^{1/\alpha}}\right] x = cx. \qquad \text{Q.E.D.}$$

We now see, combining the preceding theorems, that in the case under consideration, if the Cauchy functional equation (2) is satisfied, then the function must be $g(x) = cx$, where c is a constant.

The following statistical result may be proved using the preceding theorem.

THEOREM 3. *Let* $X_1, X_2, \cdots$ *be a sequence of non-negative independent random variables with the same continuous distribution function* $F(x)$, *and let* N_x *be defined as follows:*

$$N_x = \begin{cases} 0 \ \textit{if} \ X_1 > x, \\ n \ \textit{if} \ X_1 + X_2 + \cdots + X_n \leqq x \\ \quad \textit{and} \ X_1 + X_2 + \cdots + X_{n+1} > x. \end{cases}$$

If N_x has a distribution function of the form

$$\Pr\left\{N_x \leqq n\right\} = e^{-f(x)}\left[1 + f(x) + \frac{1}{2!}f^2(x) + \cdots + \frac{1}{[(n+1)\alpha - 1]!} f^{[(n+1)\alpha-1]}(x)\right],$$

with fixed $\alpha = 1$ or 2, for every positive x, then $F(x)$ is of the gamma type

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ \displaystyle\int_0^x \frac{\omega^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\omega/\beta} d\omega & \text{for } x \geqq 0. \end{cases}$$

Mr. Seiji Nabeya [2] has obtained a proof for the case $\alpha = 1$ when $F(x)$ is not assumed to be continuous.

REFERENCES

1. J. A Shohat and J. D. Tamarkin, *The problem of moments*, Mathematical Surveys, no. 1, New York, American Mathematical Society, 1943.

2. Seiji Nabeya, *On a relation between exponential law and Poisson's law*, Annals of the Institute of Statistical Mathematics vol. 2, no. 1 (1950).

THE UNIVERSITY OF CHICAGO