ON CHARACTERISTIC PROPERTIES OF
HARMONIC FUNCTIONS[1]

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1. **Introduction.** Green's Theorem for functions of two variables,

$$\iint_R \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dxdy + \iint_R v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dxdy$$

$$= \int_c v \frac{\partial u}{\partial n} ds,$$

(1)

yields several characteristic integral properties of harmonic functions, that is, of functions $u(x, y)$ for which

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

From (1) we obtain, for harmonic functions $u(x, y)$,

$$(2) \qquad \int_c \frac{\partial u}{\partial n} ds = 0,$$

whence the mean value properties,

$$(3) \qquad u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)d\theta,$$

$$(4) \qquad u(x_0, y_0) = \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r u(x_0 + \rho\cos\theta, y_0 + \rho\sin\theta)\rho d\rho d\theta,$$

follow easily. Equation (1) yields also

$$(5) \qquad \int_c \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds = 0$$

for pairs of harmonic functions. Further, if $u(x, y)$ is harmonic, then for each $v(x, y)$ of class C' in R, and of class C^0 in $R+C$, and satisfying $v(x, y)=0$ on C, we have, from (1),

$$(6) \qquad \iint_R \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dxdy = 0.$$

Finally, (1) yields

$$(7) \qquad \int\int_R \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy = \int_C u \frac{\partial u}{\partial n} ds$$

for harmonic functions $u(x, y)$.

Harmonic functions can be *characterized* in terms of each of the above properties. Bôcher [4] and Koebe [7] independently established the result that (2) characterizes harmonic functions. For (3) and (4), Levi [8] and Tonelli [10] proved the converse of the Gauss mean value theorem; further, the equality of the two mean value functions in (3) and (4) characterizes harmonic functions in the class of continuous functions [2; 3], and characterizes almost harmonic functions in the class of summable functions [1]. Gergen [5] showed that (5) characterizes harmonic functions $u(x, y)$ in terms of any given harmonic function $v(x, y)$ for which

$$(8) \qquad v(x, y) \neq 0.$$

And it is well known in relation to the Dirichlet principle (see, for instance [6, p. 417]), that if $u(x, y)$ is of class C' in $R + C$, and satisfies (6) for each $v(x, y)$ of class C' in $R + C$, with

$$(9) \qquad v(x, y) = 0$$

on C, then $u(x, y)$ is harmonic in R; this follows trivially from (1) if $u(x, y)$ is given to be class C'', whence the more general result can be established, for instance, by means of averaging functions.

We shall show that, under either of two additional assumptions, the first of which corresponds to Gergen's condition (8), property (7) also characterizes harmonic functions.

2. **Theorem.** Letting $o(r^\alpha)$ denote a function (not always the same function) $\phi(r)$ such that

$$\lim_{r \to 0} \frac{\phi(r)}{r^\alpha} = 0,$$

we shall establish the following result.

THEOREM 1. *If $u(x, y)$ is of class C' in a domain D, and for each (x_0, y_0) in D we have*

$$(10) \qquad \int_0^{2\pi} \int_0^r \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] \rho d\rho d\theta - \int_0^{2\pi} u \frac{\partial u}{\partial n} r d\theta = o(r^2),$$

where the integrals are taken over the area and circumference, respectively,

of the circle with center at (x_0, y_0) and radius r, and if

(11) $u(x, y) \neq 0$

in D, then $u(x, y)$ is harmonic in D.

PROOF. By hypothesis, for each point (x_0, y_0) of D we have

$$u(x, y) = u(x_0, y_0) + ar \cos \theta + br \sin \theta + o(r),$$

where a and b denote $\partial u/\partial x$ and $\partial u/\partial y$, respectively, evaluated at (x_0, y_0). Hence

$$\frac{\partial u}{\partial x} = a + o(1), \qquad \frac{\partial u}{\partial y} = b + o(1),$$

$$\frac{\partial u}{\partial n} = a \cos \theta + b \sin \theta + o(1),$$

and

$$u \frac{\partial u}{\partial n} = u(x_0, y_0) \frac{\partial u}{\partial n} + a^2 r \cos^2 \theta + 2abr \cos \theta \sin \theta + b^2 r \sin^2 \theta + o(r),$$

so that

(12) $$\int_0^{2\pi} \int_0^r \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] \rho d\rho d\theta = \pi r^2 (a^2 + b^2) + o(r^2),$$

and

(13) $$\int_0^{2\pi} u \frac{\partial u}{\partial n} r d\theta = u(x_0, y_0) \int_0^{2\pi} \frac{\partial u}{\partial n} r d\theta + \pi r^2 (a^2 + b^2) + o(r^2).$$

It follows from (10), (12), and (13) that

(14) $$u(x_0, y_0) \int_0^{2\pi} \frac{\partial u}{\partial n} r d\theta = o(r^2).$$

By Saks' generalization [9] of the above-mentioned theorem of Bôcher and Koebe, if $u(x, y)$ is of class C' in a domain, and

$$\int_0^{2\pi} \frac{\partial u}{\partial n} r d\theta = o(r^2)$$

for each point of the domain, then $u(x, y)$ is harmonic there. Accordingly, the theorem follows from (11) and (14).

3. **Lemma.** We shall use the following result.

LEMMA. *If $u(x, y)$ is of class C' in a domain D, and for each (x_0, y_0) in D either*
   (i) *there is a neighborhood of (x_0, y_0) in which $u(x, y)$ is harmonic, or*
   (ii) *$u(x_0, y_0) = 0$,*
and if $u(x, y)$ has summable second-order partial derivatives $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ in D, then $u(x, y)$ is harmonic in D.

In the lemma the assumption concerning second derivatives perhaps is redundant; but the function $u(x, y) \equiv |x|$ shows that this is not true of the assumption that $u(x, y)$ is of class C'.

PROOF OF LEMMA. We shall prove the lemma by establishing (6) for an arbitrary circular disc R, with circumference C, in D; this proof was kindly supplied by Professor L. V. Ahlfors.

Let the set of points of $R+C$ for which (ii) is satisfied be denoted by F, and let the complement of F in $R+C$ be denoted by E. For any horizontal line-segment t in $R+C$ let the intersections of t with E and F be denoted by E_t and F_t, respectively. Then F_t is closed and therefore consists of a denumerable set S_t and the set L_t of the limit points of F_t; either of the sets E_t and F_t might of course be null.

Since (ii) holds at each point of the horizontal set F_t, it follows that $\partial u/\partial x = 0$ on L_t. Further, since by hypothesis $\partial^2 u/\partial x^2$ is summable in $R+C$, it is summable on almost all horizontal line-segments t in $R+C$. Hence, integrating by parts and using the fact that (9) holds on C, we have

$$\int_t \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx = - \int_{E_t} v \frac{\partial^2 u}{\partial x^2} dx,$$

whence

$$(15) \quad \iint_R \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dA = - \int \left[\int_{E_t} v \frac{\partial^2 u}{\partial x^2} dx \right] dy = - \iint_E v \frac{\partial^2 u}{\partial x^2} dA,$$

by Fubini's theorem. Similarly,

$$(16) \quad \iint_R \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} dA = - \iint_E v \frac{\partial^2 u}{\partial y^2} dA.$$

Adding (15) and (16), we obtain (6) from the fact that (i) holds at each point of E.

4. **Theorem.** If the assumption concerning second derivatives is redundant in the lemma, then it is redundant also in the following result; in that case, condition (11) could simply be omitted in the statement of Theorem 1.

THEOREM 2. *If $u(x, y)$ is of class C' in a domain D, and (10) holds for each (x_0, y_0) in D, and if $u(x, y)$ has summable second-order partial derivatives $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ in D, then $u(x, y)$ is harmonic in D.*

PROOF. By Theorem 1, $u(x, y)$ is harmonic in the part of D where $u(x, y) \neq 0$; then Theorem 2 follows from the lemma.

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