ON THE INTEGRATION OF DIFFUSION
EQUATIONS IN RIEMANNIAN SPACES

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1. Introduction. Let R be a connected domain of an infinitely differentiable, orientable, m-dimensional ($m \geqq 2$) Riemannian space with the metric $ds^2 = g_{ij}(x)dx^i dx^j$. Under a certain "continuity condition" of Lindeberg's type, the temporally homogeneous stochastic process in R is governed by a pair of equations:[1]

$$(1.1) \qquad \frac{\partial f(x, t)}{\partial t} = b^{ij}(x) \frac{\partial^2 f(x, t)}{\partial x^i \partial x^j} + a^i(x) \frac{\partial f(x, t)}{\partial x^i}, \qquad t \geqq 0,$$

$$(1.2) \qquad \frac{\partial h(x, t)}{\partial t} = \frac{1}{(g(x))^{1/2}} \frac{\partial^2}{\partial x^i \partial x^j} ((g(x))^{1/2} b^{ij}(x) h(x, t))$$

$$- \frac{1}{(g(x))^{1/2}} \frac{\partial}{\partial x^i} ((g(x))^{1/2} a^i(x) h(x, t)), \qquad t \geqq 0.$$

These are called the "backward diffusion equation" and the "forward diffusion equation" respectively, the latter being sometimes called the Fokker-Planck's equation. In these equations, the symmetric contravariant tensor $b^{ij}(x)$ is assumed to be such that the quadratic form $b^{ij}(x)\xi_i\xi_j$ is, for $\sum_i \xi_i^2 > 0$, greater than 0 in R and $a^i(x)$ is assumed to obey, in the coordinate change $x \rightarrow \bar{x}$, the transformation rule

$$(1.3) \qquad \bar{a}^i(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k} a^k(x) + \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^s} b^{ks}(x).$$

Hence the two elliptic differential operators on the right-hand sides of (1.1) and (1.2) are formally adjoint to each other and they have a meaning independent of the local coordinates $(x^1, \cdots, x^m)$. We assume that the coefficients $g_{ij}(x)$, $a^i(x)$, and $b^{ij}(x)$ are infinitely differentiable functions of the local coordinates $(x^1, \cdots, x^m)$.

The purpose of the present note is to prove the following three theorems.

THEOREM 1. *Let R be a compact Riemannian space. Then, to any function $f(x)$, infinitely differentiable in R, there corresponds a uniquely determined solution $f(x, t)$ of* (1.1) *satisfying the conditions:*

[1] A. Kolmogoroff, *Zur Theorie der stetigen zufälligen Prozesse*, Math. Ann. vol. 108 (1933) pp. 149–160. W. Feller, *Zur Theorie der stochastischen Prozesse*, Math. Ann. vol. 113 (1937) pp. 113–160.

(1.4) $\lim_{t \downarrow 0} f(x, t) = f(x)$ *uniformly in* x,

(1.5) $\min_x f(x) \leqq f(x, t) \leqq \max_x f(x)$, *and*

$\max_x f(x, t) = \max_x f(x)$ *when* $f(x)$ *is non-negative.*

THEOREM 2. *Let R be a compact Riemannian space. Then, to any function $h(x)$, infinitely differentiable in R, there corresponds a uniquely determined solution $h(x, t)$ of (1.2) satisfying the conditions:*
(1.6) $\lim_{t \downarrow 0} \int_R |h(x, t) - h(x)| \, dx = 0$, *where* $dx = (g(x))^{1/2} dx^1 \cdots dx^m$, $g(x) = \det (g_{ij}(x))$,
(1.7) $\int_R |h(x, t)| \, dx \leqq \int_R |h(x)| \, dx$,
and $h(x, t)$ is non-negative with $\int_R h(x, t) dx = \int_R h(x) dx$ when $h(x)$ is non-negative.

If R is a connected domain with the smooth boundary ∂R, Theorem 2 is extended to the following theorem.

THEOREM 2'. *Let D be the totality of infinitely differentiable functions $h(x)$ in R with compact carriers (supports in the terminology of L. Schwartz) satisfying the boundary condition on ∂R:*

(1.8)
$$(g(x))^{1/2} b^{ij}(x) \frac{\partial h}{\partial x^i} \cos (n, x^i)$$
$$+ \left(\frac{\partial (g(x))^{1/2} b^{ij}(x)}{\partial x^i} - (g(x))^{1/2} a^j(x) \right) \cos (n, x^j) h(x) = 0,$$

(n denotes the outer normal). Then, to every $h(x) \in D$, there corresponds a uniquely determined solution $h(x, t)$ of (1.2) satisfying the conditions (1.6)–(1.7) if and only if the following hypothesis is satisfied:
THE HYPOTHESIS. *Let $\{R_k\}$ be a monotone increasing sequence of connected domains $\subseteq R$ such that the boundary ∂R_k tends smoothly, as $k \to \infty$, to the boundary ∂R. Then, for any $m > 0$, the equation*

(1.9) $$b^{ij}(x) \frac{\partial^2 f(x)}{\partial x^i \partial x^j} + a^i(x) \frac{\partial f(x)}{\partial x^i} = mf(x)$$

does not admit a bounded solution $f(x) \not\equiv 0$ satisfying the boundary condition

(1.10) $$\lim_{k \to \infty} \int_{\partial R_k} (g(x))^{1/2} b^{ij}(x) h(x) \frac{\partial f}{\partial x^i} \cos (n, x^i) dS = 0$$

for all $h(x) \in D$,

dS denoting the hypersurface element of ∂R_k.

These theorems may be proved by refining, with the aid of para-metrix considerations, the operator-theoretical integration of the dif-fusion equations given in the preceding notes.[2] It is to be remarked that our construction of the parametrix for the general diffusion equation is carried out by an elementary calculation without ap-pealing to the theory of integral equations nor to the power series expansion. It is an extension of the construction due to S. Minakshi-sundaram and Å. Pleijel.[3]

2. **The construction of the parametrix.** Let $e(x)$ be an infinitely differentiable function and let

$$(2.1) \qquad A = A_x = b^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + a^i(x)\frac{\partial}{\partial x^i} + e(x).$$

Let $\Gamma = \Gamma(P, Q) = r(P, Q)^2$ be the square of the smallest distance of two points P and Q of R according to the new metric $dr^2 = b_{ij}(x)dx^i dx^j$, where $(b_{ij}(x)) = (b^{ij}(x))^{-1}$. We have then the lemma.

LEMMA 1. *For any positive integer k, we may construct the parametrix*

$$(2.2) \quad \begin{aligned} &H_k(P, Q, t - \tau) \\ &= (t - \tau)^{-m/2} \exp\left(-\frac{\Gamma(P, Q)}{4(t - \tau)}\right)\sum_{i=0}^{k} u_i(P, Q)(t - \tau)^i, t > \tau, \end{aligned}$$

such that

(2.3) $u_i(P, Q)$ $(i = 0, 1, \cdots, k)$ *are infinitely differentiable in the vicinity of $Q = P$ and $u_0(P, P) = 1$,*

$$(2.4) \quad \begin{aligned} &\left(-\frac{\partial}{\partial\tau} - A_Q\right)H_k(P, Q, t - \tau) \\ &\qquad\qquad = (t - \tau)^{k - m/2}\exp\left(-\frac{\Gamma(P, Q)}{4(t - \tau)}\right)c_k(P, Q), \end{aligned}$$

where $c_k(P, Q)$ is infinitely differentiable in the vicinity of $Q = P$.

[2] K. Yosida, *Integration of Fokker-Planck's equation in a compact Riemannian space*, Arkiv. för Matematik vol. 1 (1949) pp. 71–75 (to be referred to as [I]). K. Yosida, *Integration of Fokker-Planck's equation with a boundary condition*, Journal of the Mathematical Society of Japan vol. 3 (1951) pp. 69–73 (to be referred to as [II]). K. Yosida, *Integrability of the backward diffusion equation in a compact Riemannian space*, Nagoya Math. J. vol. 3 (1951) pp. 1–4 (to be referred to as [III]).

[3] *Some properties of the eigenfunctions of the Laplace operator on Riemannian mani-folds*, Canadian Journal of Mathematics vol. 1 (1949) pp. 242–256.

PROOF. We introduce the normal coordinates y^σ of $Q = (x^1, \cdots, x^m)$ in the vicinity of P:

$$(2.5) \qquad y^\sigma = (\Gamma(P, Q))^{1/2} \left(\frac{dx^\sigma}{dr} \right)_{Q=P}.$$

Let

$$(2.6) \qquad dr^2 = \beta_{ij}(y) dy^i dy^j.$$

We have the well known formulae

$$(2.7) \qquad \Gamma(P, Q) = \beta_{ij}(0) y^i y^j, \qquad \beta_{ij}(y) y^j = \beta_{ij}(0) y^j.$$

By virtue of (2.7), the operator

$$(2.8) \qquad A = A_y = \beta^{ij}(y) \frac{\partial^2}{\partial y^i \partial y^j} + \alpha^i(y) \frac{\partial}{\partial y^i} + e(y),$$

$$(\beta^{ij}(y)) = (\beta_{ij}(y))^{-1},$$

when applied to the function $f(\Gamma, y)$, where Γ is considered as a function of y, may be written as follows:

$$(2.9) \qquad A_y f = 4\Gamma \frac{\partial^2 f}{\partial \Gamma^2} + 4y^\sigma \frac{\partial^2 f}{\partial \Gamma \partial y^\sigma} + M \frac{\partial f}{\partial \Gamma} + N(f),$$

where

$$M = \beta^{ij} \frac{\partial^2 \Gamma}{\partial y^i \partial y^j} + \alpha^i \frac{\partial \Gamma}{\partial y^i} = 2m + O(y),$$

$$N(f) = \beta^{ij} \frac{\partial^2 f}{\partial y^i \partial y^j} + \alpha^i \frac{\partial f}{\partial y^i} + ef.$$

The differentiations in $A_y f$ and $N(f)$ are to be performed as if Γ and y are independent variables. Hence we have

$$- A_y H_k = \sum_{i=0}^{k} - \frac{\Gamma}{4} (t - \tau)^{i-2-m/2} \exp\left(- \frac{\Gamma}{4(t - \tau)} \right)$$

$$+ \sum_{i=0}^{k} (t - \tau)^{i-1-m/2} \exp\left(- \frac{\Gamma}{4(t - \tau)} \right) \left\{ y^\sigma \frac{\partial u_i}{\partial y^\sigma} \right.$$

$$\left. + \frac{M}{4} u_i - N(u_{i-1}) \right\}$$

$$+ (t - \tau)^{k-m/2} \exp\left(- \frac{\Gamma}{4(t - \tau)} \right) N(u_k),$$

where $u_{-1} \equiv 0$ and hence $N(u_{-1}) \equiv 0$. Therefore, since

$$-\frac{\partial}{\partial \tau} H_k = \sum_{i=0}^{k} (t - \tau)^{i-1-m/2} \exp\left(-\frac{\Gamma}{4(t-\tau)}\right)$$
$$\cdot u_i\left(\frac{-m}{2} + i + \frac{\Gamma}{4(t-\tau)}\right),$$

we obtain Lemma 1 if the u_i are successively so chosen that

$$(2.10) \qquad y^\sigma \frac{\partial u_i}{\partial y^\sigma} + \left(\frac{-m}{2} + i + \frac{M}{4}\right) u_i = N(u_{i-1}),$$

$u_i(P, Q)$ being infinitely differentiable in the vicinity of $Q = P$ with $u_{-1} \equiv 0$ and $u_0(P, P) = 1$. Such u_i may be determined, in view of the order relation

$$(2.11) \qquad M = 2m + O(y).$$

For this purpose, put $y^\sigma = \eta^\sigma s$ and transform (2.10) into ordinary differential equations in s containing the parameters η. The equations are integrated by

$$(2.12) \qquad \begin{aligned} u_0(P, Q) &= \exp\left(-\int_0^s s^{-1}\left(\frac{-m}{2} + \frac{M}{4}\right) ds\right), \\ u_i(P, Q) &= u_0 s^{-i} \int_0^s s^{i-1} u_0^{-1} N(u_{i-1}) ds \qquad (i = 1, \cdots, k). \end{aligned}$$

REMARK. By (2.9) and (2.10), we have

$$(2.13) \qquad \begin{aligned} A_Q u_0(P, Q)\Gamma(P, Q)^{(2-m)/2} &= N(u_0(P, Q))\Gamma(P, Q)^{(2-m)/2}, \quad m \geqq 3, \\ A_Q u_0(P, Q) \log \Gamma(P, Q) &= N(u_0(P, Q)) \log \Gamma(P, Q), \quad m = 2. \end{aligned}$$

We have thus obtained the parametrix for the elliptic differential operator A.

3. The proof of Theorem 2'. Let $e \equiv 0$ in the operator A and let

$$(3.1) \qquad \begin{aligned} (A'h)(x) &= \frac{1}{(g(x))^{1/2}} \frac{\partial^2}{\partial x^i \partial x^j} ((g(x))^{1/2} b^{ij}(x) h(x)) \\ &\quad + \frac{1}{(g(x))^{1/2}} \frac{\partial}{\partial x^i} (-(g(x))^{1/2} a^i(x) h(x)) \end{aligned}$$

be the formally adjoint operator of A. Let $L_1(R)$ be the Banach space of the totality of the functions $h(x)$ integrable with respect to dx in

R, metrized by the norm $\|h\| = \int_R |h(x)| \, dx$. D is a dense linear subset of $L_1(R)$. We take the operator A' to be an additive operator defined on $D \subseteq L_1(R)$ into $L_1(R)$, and let $\tilde{A}'$ be the smallest closed extension of the operator A'. Then[4] there exists a uniquely determined one-parameter semi-group[5] of linear operators T_t on $L_1(R)$ into $L_1(R)$ satisfying the conditions

$$T_t T_s = T_{t+s} \ (t, s) \geqq 0), \ T_0 = \text{the identity},$$

$$\text{strong} \lim_{t \to t_0} T_t h = T_{t_0} h \quad \text{for } h \in L_1(R),$$

(3.2) $(T_t h)(x)$ is non-negative and $\|T_t h\| = \|h\|$ when $h(x)$ is non-negative,

$$\partial_t T_t h = \text{strong} \lim_{\delta \to 0} \frac{T_{t+\delta} h - T_t h}{\delta} = \tilde{A}' T_t h \text{ for } h \text{ in the domain}$$

$\tilde{D}'$ of the operator $\tilde{A}'$,

if and only if the hypothesis of Theorem 2′ is satisfied.

We thus have to show that the function $h(x, t) = (T_t h)(x)$ is equivalent (in the sense to be explained below) to a function which is continuously differentiable once in t and twice in x.

For this purpose we prepare two lemmas.

LEMMA 2. *Let $f(x, t)$ be an infinitely differentiable function which vanishes outside a compact coordinate neighborhood of P. Then*

$$\int_R h(y, t) f(y, t) dy$$

(3.3) $$= \int_R h(y, 0) f(y, 0) dy$$

$$+ \int_0^t d\tau \int_R \left\{ \partial_\tau h(y, \tau) f(y, \tau) + h(y, \tau) \frac{\partial f(y, \tau)}{\partial \tau} \right\} dy.$$

PROOF. By (3.2), $h(y, \tau)$ is strongly differentiable in τ with the differential quotient $\tilde{A}'_y h(y, \tau)$. Hence $h(y, \tau) f(y, \tau)$ is weakly differ-

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[4] See [II]. Cf. also [I]. The hypothesis is surely satisfied when R is a connected domain in m-dimensional Euclidean space whose boundary lies entirely in the compact part of the space and when, moreover, the operator A is the usual Laplacian. See K. Yosida, *A theorem of Liouville's type for meson equation*, Proc. Imp. Acad. Tokyo vol. 27 (1951) pp. 214–215.

[5] E. Hille, *Functional analysis and semi-groups*, New York, 1948. Cf. also K. Yosida, *On the differentiability and the representation of one-parameter semi-group of linear operators*, Journal of the Mathematical Society of Japan vol. 1 (1948) pp. 15–21.

entiable in τ with the differential quotient

$$f(y, \tau)\tilde{A}'_y h(y, \tau) + \frac{\partial f(y, \tau)}{\partial \tau} h(y, \tau)$$

which is strongly continuous in τ. Thus, by integration, we obtain (3.3).

LEMMA 3. *We have*

$$\int_R h(y, t)f(y, t)dy = \int_R h(y, 0)f(y, 0)dy$$

(3.4)

$$= \int_0^t d\tau \int_R \left\{ h(y, \tau)\left(\frac{\partial f(y, \tau)}{\partial \tau} + A_y f(y, \tau) \right) \right\} dy.$$

PROOF. The right-hand side is, by (3.2) and (3.3), equal to

$$-\int_0^t d\tau \int_R \left\{ h(y, \tau)\left(-\frac{\partial f(y, \tau)}{\partial \tau} - A_y f(y, \tau) \right) \right.$$

$$\left. - f(y, \tau)(\partial_\tau h(y, \tau) - \tilde{A}'_y h(y, \tau)) \right\} dy$$

$$= \left[\int_R h(y, \tau)f(y, \tau)dy \right]_0^t$$

$$+ \int_0^t d\tau \int_R \left\{ h(y, \tau)A_y f(y, \tau) - f(y, \tau)\tilde{A}'_y h(y, \tau) \right\} dy.$$

That the second integral on the right-hand side is equal to zero may be seen by the following argument. Let $\{h_k(y, \tau)\} \subseteq D$ be such that strong $\lim_{k\to\infty} h_k(y, \tau) = h(y, \tau)$, strong $\lim_{k\to\infty} A'_y h_k(y, \tau) = \tilde{A}'_y h(y, \tau)$. The existence of such a sequence $\{h_k(y, \tau)\}$ is assured by the fact that $h(y, \tau) = (T_\tau h)(y)$ is in the domain $\tilde{D}'$ of the operator $\tilde{A}'$ which is the smallest closed extension of the operator A' with the domain D. We have thus

$$\int_R \left\{ h(y, \tau)A_y f(y, \tau) - f(y, \tau)\tilde{A}'_y h(y, \tau) \right\} dy$$

$$= \lim_{k\to\infty} \int_R \left\{ h_k(y, \tau)A_y f(y, \tau) - f(y, \tau)A'_y h_k(y, \tau) \right\} dy.$$

The right-hand integrals are equal to zero as may be seen by Green's integral theorem and the fact that $f(y, \tau)$ vanishes outside a compact coordinate neighborhood of P.

After these preliminaries, we may give:

PROOF OF THE THEOREM 2'. We shall apply (3.4) to the case when

$$(3.5) \qquad f(y, \tau) = f(Q, \tau) = H_k(P, Q, t + \epsilon - \tau)\delta(P, Q)\delta(P_0, P).$$

Here P_0 is an arbitrary point of R, fixed in the following argument, ϵ is a positive constant, and

$$(3.6) \qquad\qquad \delta(P, Q) = \alpha(r(P, Q)),$$

where $\alpha(r)$ denotes an infinitely differentiable function of r satisfying

$$\alpha(r) = \begin{cases} 1 & \text{for} \quad r \leqq 2^{-1}\eta, \\ \text{between 0 and 1} & \text{for} \quad 2^{-1}\eta < r < \eta, \\ 0 & \text{for} \quad r \geqq \eta. \end{cases}$$

We assume that the positive constant η is chosen so small that
   (3.7) the points Q satisfying $\delta(P_0, P)\delta(P, Q) > 0$ are contained in a compact coordinate neighborhood of P_0.
   We have thus

$$\int_R h(Q, t)H_k(P, Q, \epsilon)\delta(P_0, P)\delta(P, Q)dy$$

$$(3.8) \qquad\qquad - \int_R h(Q, 0)H_k(P, Q, t + \epsilon)\delta(P_0, P)\delta(P, Q)dy$$

$$= - \int_0^t d\tau \int_R h(Q, \tau)K_k(P, Q, t + \epsilon - \tau)dy,$$

where

$$(3.9) \qquad \begin{aligned} & K_k(P, Q, t + \epsilon - \tau) \\ & = \left(-\frac{\partial}{\partial\tau} - A_Q\right)(H_k(P, Q, t + \epsilon - \tau)\delta(P_0, P)\delta(P, Q)). \end{aligned}$$

Let k be taken so large that

$$(3.10) \qquad\qquad \frac{-m}{2} + k \geqq 2.$$

Then, in view of Lemma 1, $K_k(P, Q, t+\epsilon-\tau)$ is, for $r(P_0, P) \leqq 2^{-1}\eta$, devoid of the singularity even if $(t+\epsilon-\tau) = 0$. We shall next show that the first term on the left-hand side of (3.8) tends, as $\epsilon \downarrow 0$, strongly to $h(P, t)$ in the vicinity of P_0. This may be proved as follows. We have, by (3.6) and (3.7),

$$\int_R \delta(P_0, P) dP \left| \int_R h(Q, t) H_k(P, Q, \epsilon) \delta(P, Q) dy \right.$$

$$\left. - h(P, t) \int_R H_k(P, Q, \epsilon) \delta(P, Q) dy \right|$$

$$\leqq C \int_{r(P_0, Q) \leqq 2\eta} \left(\int_{r(P_0, P) \leqq \eta} |h(Q, t) - h(P, t)| dP \right) |H_k(P, Q, \epsilon)| dy$$

$$\leqq C_1 \int \cdots \int \left(\int |h(z + \epsilon^{1/2}\xi, t) - h(z, t)| dz \right)$$

$$\cdot \exp \left(- \sum_i (\xi^i)^2 \right) d\xi^1 \cdots d\xi^m.$$

Here $(z^1, \cdots, z^m)$ and $(z^1 + y^1, \cdots, z^m + y^m)$ are the coordinates of the points P and Q respectively in the coordinate neighborhood of P_0, defined in (3.7), and C and C_1 denote suitable positive constants. The inner integral on the right-hand side converges, when $\epsilon \downarrow 0$, to zero boundedly by Lebesgue's theorem.

Therefore, there exists a sequence $\{\epsilon_i\}$ with $\epsilon_i \downarrow 0$ such that

$$h(P, t) \lim_{i \to \infty} \int_R H_k(P, Q, \epsilon_i) \delta(P, Q) dy$$

$$(3.11) \qquad = \int_R h(Q, 0) H_k(P, Q, t) \delta(P, Q) dy$$

$$- \int_0^t d\tau \int_R h(Q, \tau) K_k(P, Q, t - \tau) dy$$

almost everywhere with respect to P in the vicinity of P_0. Hence, by (3.10)–(3.11), $h(P, t)$ may be considered to be continuously differentiable once in $t > 0$ and twice in P in the vicinity of P_0 if

$$(3.12) \quad \lim_{\epsilon \downarrow 0} \int_R H_k(P, Q, \epsilon) \delta(P, Q) dy \text{ is positive and twice continuously}$$

differentiable in P in the vicinity of P_0.

The proof of (3.12) may be obtained as follows. We have

$$\lim_{\epsilon \downarrow 0} \int_R H_k(P, Q, \epsilon) \delta(P, Q) dy$$

$$= \lim_{\epsilon \downarrow 0} \int_{r(P, Q) \leqq \zeta} \epsilon^{-m/2} \exp \left(- \frac{\Gamma(P, Q)}{4\epsilon} \right) dy$$

for any positive constant ζ. Hence, by putting

$$(3.13) \qquad ds^2 = \gamma_{ij}(y)dy^i dy^j, \qquad y^i = \epsilon^{1/2}\xi^i,$$

we obtain, in view of the arbitrariness of ζ,

$$(3.14)
\begin{aligned}
&\lim_{\epsilon \downarrow 0} \int_R H_k(P, Q, \epsilon)\delta(P, Q)dy \\
&= \lim_{\epsilon \downarrow 0} \int \cdots \int_{-\zeta \leq \epsilon^{1/2}\xi^i \leq \zeta} \exp\left(-\beta_{ij}(0)\xi^i\xi^j\right)(\gamma(0))^{1/2}d\xi^1 \cdots d\xi^m \\
&= \pi^{m/2}(\gamma(0))^{1/2}(\beta(0))^{1/2} = \pi^{m/2}(g(P))^{1/2}/(b(P))^{1/2},
\end{aligned}$$

where

$$g(P) = \det\left(g_{ij}(P)\right) \quad \text{and} \quad b(P) = \det\left(b_{ij}(P)\right).$$

4. The proof of Theorem 1. Let $C(R)$ be the Banach space of the totality of continuous functions $f(x)$ in R, metrized by the norm $\|f\| = \max_x |f(x)|$. D is a dense linear subset of $C(R)$. Let $e \equiv 0$ in the operator A. We consider this A to be an additive operator defined on $D \subseteq C(R)$ into $C(R)$, and define the smallest closed extension $\tilde{A}$ of A. Then[6] there exists a uniquely determined one-parameter semi-group of linear operators S_t on $C(R)$ into $C(R)$ satisfying the conditions:

$$(4.1)
\begin{aligned}
&S_t S_s = S_{t+s} \quad (t, s \geq 0), \qquad S_0 = \text{the identity } I, \\
&\text{strong } \lim_{t \to t_0} S_t f = S_{t_0} f \quad \text{for} \quad f \in C(R), \\
&(S_t f)(x) \text{ is non-negative and } \max_x (S_t f)(x) = \max_x f(x) \text{ if } f(x) \\
&\text{is non-negative} \\
&\partial_t S f = \text{strong } \lim_{\delta \to 0} \frac{S_{t+\delta}f - S_t f}{\delta} = \tilde{A}S_t f \text{ for } f \text{ in the domain } \tilde{D} \text{ of}
\end{aligned}$$

the operator A.

Therefore, as in §3, we may prove that $f(x, t) = (S_t f)(x)$ may be considered to be continuously differentiable once in t and twice in x.

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[6] See [III].