A REMARK ON M. M. DAY'S CHARACTERIZATION OF INNER-PRODUCT SPACES AND A CONJECTURE OF L. M. BLUMENTHAL[1]

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1. A space of elements $a, b, \cdots$, with a distance function ab is said to be *semimetric* provided $ab = ba > 0$ if $a \neq b$, and $aa = 0$. A real-linear space of elements $f, g, \cdots$ is said to be *semi-normed* provided a function $\|f\|$ is defined in S having the usual properties of a norm with the exception of the inequality $\|f+g\| \leqq \|f\| + \|g\|$, which is not assumed. Evidently $\|f-g\|$ is a semimetric in the sense of the first definition.

A semimetric space is called *ptolemaic* provided that among the distances between any four points a, b, c, d Ptolemy's inequality

$$ab \cdot cd + ad \cdot bc \geqq ac \cdot bd \tag{1}$$

always holds. It is known that a real inner-product space is ptolemaic.[2] Recently L. M. Blumenthal has orally raised the question as to the validity of the converse proposition in the following sense: Let the real normed space S be ptolemaic; does it follow that its norm springs from an inner product? His conjecture in the affirmative is verified in a somewhat more general setting by the following theorem.

THEOREM 1. *Let S be a real semi-normed space which is ptolemaic. Then $\|f\|$ is a norm which springs from an inner product, i.e., S is a real inner-product space.*

2. This theorem is closely related to the characterizations of inner-product spaces among normed linear spaces. It was shown by Jordan and von Neumann [2] that a normed linear space S is an inner-product space if and only if we have the identity

$$\|f - g\|^2 + \|f + g\|^2 = 2\|f\|^2 + 2\|g\|^2 \qquad (f, g \in S). \tag{2}$$

M. M. Day has shown (Theorem 2.1 [1]) that S is an inner-product space if we require only that (2) holds for f and g on the unit sphere. In other words, he has shown that (2) may be replaced by the condition

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[2] See [3, p. 716], in the list of references at the end of this note.

$$\|f-g\|^2 + \|f+g\|^2 = 4, \qquad (\|f\| = 1, \|g\| = 1). \tag{3}$$

I wish to point out now that Day's condition (3) may be weakened still further as stated by the following theorem.

THEOREM 2. *The real normed space S is an inner-product space if it has the property that*

$$\|f-g\|^2 + \|f+g\|^2 \geqq 4, \qquad (\|f\| = 1, \|g\| = 1). \tag{4}$$

PROOF.[3] As in all characterizations of inner-product spaces, it suffices to assume that S is 2-dimensional, and hence is a Minkowskian plane with a gauge curve

$$\Gamma: \ \|f\| = 1$$

which is convex and has the origin 0 as center. The problem now amounts to showing that Γ is an ellipse. Let f and g be two points on $\Gamma(f \neq \pm g)$ and let us see what the inequality (4) means in geometrical terms. Consider the parallelogram of vertices f, g, $-f$, $-g$. Draw the two diameters of Γ that are parallel to the sides joining f to g and f to $-g$, and denote their euclidean half-lengths by α and β, respectively. Let (x, y) be the oblique coordinates of the point f in the system formed by these diameters. We now find that

$$\|f-g\| = 2\,|\,x\,|\,/\alpha, \qquad \|f+g\| = 2\,|\,y\,|\,/\beta,$$

which allow us to rewrite (4) as

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \geqq 1. \tag{5}$$

The condition (4) amounts therefore to the following geometric property of the curve Γ: *If AA' and BB' are any two distinct diameters of Γ and MM' and NN' are its diameters parallel to AB and AB', respectively, then none of the points A, B, A', B' are ever inside the ellipse having MM' and NN' as conjugate diameters.*

Let us assume now that by some means we have found an ellipse E with center 0, enjoying the following properties: (i) No point of Γ is inside E, (ii) E and Γ have the distinct pairs of opposite points A, A', B, B' in common. We claim now that E and Γ must coincide. Indeed, draw the diameters MM' and NN' of Γ as above. Then E must

[3] Our proof of Theorem 2 is implicitly contained in Day's elegant proof of the sufficiency of (3). His proof actually establishes the sufficiency of the weaker condition (3') $\|f-g\|^2+\|f+g\|^2 \leqq 4$ $(\|f\|=1, \|g\|=1)$. In dealing with (4) we apply Day's procedure "from the inside out," as Day himself does in another connection (See [1, p. 328, proof of Theorem 4.2]).

pass through their end points M, M', N, N', otherwise the ellipse E_1 of conjugate diameters MM', and NN', which evidently contains E, would contain the four points A, B, A', B' inside, which contradicts the property of Γ derived from (4). The process may now be repeated with any of the two pairs like M, M', A, A', leading to four new and distinct pairs of points of E which are common with Γ. We reach in this way common points of E and Γ which are evidently dense on E and the identity between E and Γ follows.

There still remains to show how to obtain an ellipse E with the properties (i), (ii) used above. Let E be an ellipse[4] of center 0, inscribed in Γ, and having the maximal area among all such ellipses. We claim that E enjoys the properties (i), (ii). Indeed, let us assume this not to be the case; rather let Γ and E have only the points A and A' in common. An affine transformation shows that we lose no generality by assuming E to be the circle $x^2+y^2=1$, $A=(1, 0)$, $A'=(-1, 0)$. Consider now the one-parameter family of ellipses $x^2/a^2+y^2/b^2=1$ passing through the four fixed points $(\pm 1/2^{1/2}, \pm 1/2^{1/2})$. Among them the circle E has least area. If a is less than 1 and sufficiently close to 1, it is clear that the corresponding ellipse is wholly inside Γ, which contradicts the maximal area property of the circle E.

3. We are now able to prove Theorem 1 in a few lines. Let us first show that the *semi-norm* $\|f\|$ is a *norm*, i.e., satisfies

$$\|f+g\| \leqq \|f\| + \|g\|. \tag{6}$$

Applying Ptolemy's inequality (1) to the points

$$a = 0, \quad b = f, \quad c = (f+g)/2, \quad d = g \qquad (f \neq g),$$

we find that

$$\|f\|\cdot\left\|\frac{f-g}{2}\right\| + \|g\|\cdot\left\|\frac{f-g}{2}\right\| \geqq \left\|\frac{f+g}{2}\right\|\cdot\|f-g\|$$

and dividing this inequality by $\|f-g\|/2$ we find that

$$\|f\| + \|g\| \geqq \left\|\frac{f+g}{2}\right\|\cdot 2 = \|f+g\|$$

which proves (6). Thus S is a real normed space.

Applying again Ptolemy's inequality (1) to the points

$$a = f, \qquad b = g, \qquad c = -f, \qquad d = -g,$$

[4] Its existence is clear; its unicity is irrelevant for our purpose.

we obtain

$$\|f-g\|^2 + \|f+g\|^2 \geqq 4\|f\|\cdot\|g\| \qquad (f, g \in S).^5 \tag{7}$$

This plainly implies (4) and now S is an inner-product space by Theorem 2.

4. The ptolemaic inequality (1) was introduced in [3] in order to formulate a result of Menger in the following improved form: *A simple metric arc γ is congruent to a segment if and only if* (α)γ *has vanishing Menger curvature in all its points,* (β) *Ptolemy's inequality holds throughout* γ.

In view of this result, Theorem 1 now suggests the following question: *Let γ be a simple arc in a linear normed space S with the property that γ has vanishing Menger curvature in all its points. For which spaces S, other than inner-product spaces, is it true that γ is congruent to a segment?*

That the answer is not unconditionally affirmative is shown by the following counter-example due to L. M. Blumenthal: Let S be the 2-dimensional space of points $f=(x, y)$ with the norm $\|f\| = |x| + |y|$. Let the arc γ be the polygonal line of successive vertices (0, 1), (0, 0), (1, 0), (1, 1). γ is seen to be "locally straight," hence of vanishing curvature in all its points. However, the distance between its end points is equal to 1, which is different from the sum 3 of the lengths of its three component segments. The arc γ is therefore not congruent to a segment.

References

1. M. M. Day, *Some characterizations of inner-product spaces*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 320–337.

2. P. Jordan and J. von Neumann, *On inner products in linear metric spaces*, Ann. of Math. (2) vol. 36 (1935) pp. 719–723.

3. I. J. Schoenberg, *On metric arcs of vanishing Menger curvature*, Ann. of Math. (2) vol. 41 (1940) pp. 715–726.

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[5] It is interesting to notice the equivalence of the conditions (2) and (7). Clearly (2) implies (7) formally; that (7) implies (2) is just being shown.