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ARBITRARY MAPPINGS

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The results of this paper generalize some results of H. Blumberg.[1]

Let X be a set of elements and let $\mathfrak{N}$ be a collection of nonempty subsets of X. We assume that there is, in $\mathfrak{N}$, a countable subcollection: $N^1, N^2, \cdots, N^n, \cdots$ with the property that, for each N of $\mathfrak{N}$ and each x of N, there is an integer k, such that $x \in N^k$ and $N^k \subseteq N$. In the remainder of this paper the letter N with a superscript will always denote a member of the set $N^1, N^2, \cdots, N^n, \cdots$. The letter N with or without subscripts will always denote a member of $\mathfrak{N}$, and the letter x will denote an element of X. The symbol $N(x)$ denotes a set N which contains x.

Let the statements of the preceding paragraph be repeated, replacing X by Y, $\mathfrak{N}$ by $\mathfrak{M}$, N by M, x by y.

Let a correspondence, f, be given which to each x assigns a nonempty subset (denoted by $f(x)$) of Y. If $V \subseteq Y$, then $f^{-1}(V)$ denotes the set of all x such that $f(x) \cdot V \neq 0$.

DEFINITIONS. *Let S be a subset of X. Then S is nowhere dense* if, for each N, there is an $N_1 \subseteq N$ with $N_1 \cdot S = 0$. A set is *exhaustible* if it is the union of a countable collection of nowhere dense sets. A set is *residual* if it is the complement (with respect to X) of an exhaustible set. A set is *inexhaustible* if it is not exhaustible. The point x is said to be a *point of exhaustible f-approach* if there is a y in $f(x)$, an $M(y)$, and an $N(x)$ such that $[f^{-1}(M(y))] \cdot N(x)$ is exhaustible.

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[1] *New properties of all real functions*, Trans. Amer. Math. Soc. vol. 24 (1922) pp. 113–128. Also *Arbitrary point transformations*, Duke Math. J. vol. 11 (1944) pp. 671–685. The definition of f-approach and λ approach that we use here are essentially those introduced by Blumberg.

If x is not a point of exhaustible f-approach, then it is said to be a *point of inexhaustible f-approach.*

LEMMA. *The points of exhaustible f-approach form an exhaustible set.*

PROOF. Let A be the set of points of exhaustible f-approach and let x be an element of A. Then there is a y in $f(x)$, an $M(y)$, and an $N(x)$ such that $[f^{-1}(M(y))]\cdot N(x)$ is exhaustible. Select $M^k(y)\subseteq M(y)$ and $N^p(x)\subseteq N(x)$. Let $A_{k,p}$ be the subset of X given by: $A_{k,p} = [f^{-1}(M^k(y))]\cdot[N^p(x)]$. Then $x\in A_{k,p}$ and $A_{k,p}$ is exhaustible. Associate with each x in A such a pair of integers $k(x)$, $p(x))$ and let U be the union of the sets $A_{k(x),p(x)}$ thus defined. Then $A\subseteq U$. Since U is exhaustible, so is A.

DEFINITION. The point x is said to be a *point of concentrated inexhaustible f-approach* if for each y in $f(x)$ and each $M(y)$ there is an $N(x)$ such that for every $N_\alpha\subseteq N(x)$, the set $[f^{-1}(M(y))]\cdot N_\alpha$ is inexhaustible.

THEOREM. *The points of concentrated inexhaustible f-approach form a residual set.*

PROOF. The proof will be based on the following lemma. In this way we illustrate a method of procedure which might also have been used in the proof of the preceding lemma.

LEMMA. *Let λ be a set property, defined on all of $\mathfrak{N}$. The point x is said to be a point of λ approach if each $N(x)$ has the property λ. The point x is said to be a point of concentrated λ approach if there is an $N(x)$ such that any $N_\alpha\subseteq N(x)$ has the property λ. Then the set of points which are of λ approach but are not of concentrated λ approach is nowhere dense.*

The proof is immediate.

We proceed with the proof of the theorem. Let λ_k be the property of N that $[f^{-1}(M^k)]\cdot N$ is inexhaustible. Let B be the set of points of inexhaustible, but not concentrated inexhaustible, f-approach. Let x be an element of B. For any k such that $f(x)\cdot M^k\neq 0$, x is a point of λ_k approach (since x is of inexhaustible f-approach). On the other hand (since x is not a point of concentrated inexhaustible f-approach), there is some integer, p, such that: (i) $f(x)\cdot M^p\neq 0$, and (ii) x is not a point of concentrated λ_p approach. Associate with each $x\in B$ such a $p(x)$ and let B_m be the union of all $x\in B$ thus associated with the integer m. Each $x\in B_m$ is then λ_m approached, but not concentrated λ_m approached. By the lemma, B_m is nowhere dense, and, by construction, $B=\sum_m B_m$. Hence B is exhaustible. Since the set of points of concentrated inexhaustible f-approach is $X-(A+B)$, it is residual.

In order to prove the corollaries that follow regarding continuity, we shall assume for the remainder of this paper that, in addition to the assumptions already made regarding $(X, \mathfrak{N})$, $(Y, \mathfrak{M})$, and f, the following conditions are also satisfied: (1) f is single point valued; i.e., for each x, $f(x)$ is a single element of Y. (2) For any $M_1(y)$ and $M_2(y)$ there is an $M_3(y) \subseteq M_1(y) \cdot M_2(y)$. (3) X is a Hausdorff space having the collection $\mathfrak{N}$ as a neighborhood system; i.e. each $N(x)$ is a neighborhood of x in the Hausdorff sense. (4) Each N is an inexhaustible set.

We also make some further definitions. If $S \subseteq X$, then the closure of S is denoted by cS. A *set of separated points* is a set, S, such that for each x in S there is an $N(x)$ such that $c[N(x)] \cdot (S - \{x\}) = 0$. The function f_S is defined by $f_S(x) = f(x)$ for $x \in S$ and is undefined for $x \notin S$. We say that f_S is *continuous* at $x \in S$ if, for each $M(f(x))$, there is an $N(x)$ such that, for every $\xi \in N(x) \cdot S$, $f(\xi) \in M(f(x))$. If f_S is continuous at each $x \in S$, then f_S is said to be continuous. The point $x \in S$ is said to be a point of *concentrated inexhaustible f_S-approach* if, for each $M(f(x))$, there is an $N(x)$ such that, for every $N_\alpha \subseteq N(x)$, $[f^{-1}(M(f(x)))] \cdot N_\alpha \cdot S$ is inexhaustible. Finally let $\mathfrak{I}$ denote the class of subsets of X defined as follows: S belongs to $\mathfrak{I}$ if and only if: (i) for each N, $N \cdot S$ is inexhaustible (i.e., S is of the second Baire category at each point, and (ii) each $x \in S$ is of concentrated inexhaustible f_S-approach.

Corollary 1. *The class $\mathfrak{I}$ is not empty* (for by condition (4) the residual set of the theorem belongs to $\mathfrak{I}$).

Corollary 2. *Let I_1 belong to $\mathfrak{I}$ and let T be an arbitrary set of separated points belonging to I_1. Then there is an I_2 in $\mathfrak{I}$ such that $T \subseteq I_2 \subseteq I_1$ and f_{I_2} is continuous at each $x \in T$.*

Proof. First suppose that T consists of a single point, ξ. If T is an $N(\xi)$, the proof is complete with $I_2 = I_1$. If T is not any $N(\xi)$, then it follows from condition (3) that there is a sequence of properly decreasing sets $N^{k_1}(\xi)$, $N^{k_2}(\xi)$, $\cdots$, $N^{k_n}(\xi)$, $\cdots$ such that, for each n, $N^{k_n} - N^{k_{n+1}}$ contains an N and, for each $N(\xi)$, there is an integer p with $N^{k_p}(\xi) \subseteq N(\xi)$. Say $f(\xi) = \eta$. If η is in no M, the proof is complete with $I_2 = I_1$. If there is an $M(\eta)$, then by condition (2) we can take a sequence of nonincreasing sets: $M^{\gamma_1}(\eta)$, $M^{\gamma_2}(\eta)$, $\cdots$, $M^{\gamma_n}(\eta)$, $\cdots$ (possibly terminating), such that for any $M(\eta)$ there is an integer m with $M^{\gamma_m} \subseteq M(\eta)$.

Now there is an $N^{k_p}(\xi)$ such that, for any $N \subseteq N^{k_p}(\xi)$, $[f^{-1}(M^{\gamma_1})] \cdot N \cdot I_1$ is inexhaustible. Let S_1 be the set of all x such that $f(x) \in M^{\gamma_1}$

and let $K_1 = I_1 - (cN^{k_p} - S_1 \cdot cN^{k_p}) \cdot I_1$. Then K_1 belongs to $\mathfrak{J}$. If there is no $M^{\gamma_2}(\eta)$, we may stop. If there is one, then there is an $N^{k_q}(\xi)$ such that $q > p$ and, for any $N \subseteq N^{k_q}(\xi)$, $[f^{-1}(M^{\gamma_2})] \cdot N \cdot K_1$ is inexhaustible, since $M^{\gamma_2} \subseteq M^{\gamma_1}$. Take S_2 to be the set of all x such that $f(x) \in M^{\gamma_2}$ and let $K_2 = K_1 - (cN^{k_q} - S_2 \cdot cN^{k_q}) \cdot K_1$. Then K_2 belongs to $\mathfrak{J}$. Constructing, in this manner, a sequence $\{K_n\}$, we see that the set $I_2 = \prod_n K_n$ has the properties stated.

If T is an arbitrary set of separated points, we take, for each x in T, a neighborhood $N_x(x)$ whose closure is disjoint from $T - \{x\}$ and carry out the same construction, starting the sequence of descending neighborhoods, $\{N_x^{k_n}\}$, about each x of T, with the neighborhood $N_x(x)$.

Corollary 3. *Let I_1 belong to $\mathfrak{J}$ and let F be any set consisting of a finite number of points in I_1. Then there exists an everywhere dense set, D, with $F \subseteq D \subseteq I_1$, such that f_D is continuous.*

Proof. Select ξ_1 arbitrarily from $N^1 \cdot I_1$. Apply Corollary 2 (replacing T there by $F + \{\xi_1\}$), obtaining the set I_2. Proceed in this manner; viz., at the jth step take ξ_j arbitrarily from $N^j \cdot I_j$ and apply Corollary 2 (replacing I_1 there by I_j, T by $(F + \{\xi_1\} + \{\xi_2\} + \cdots + \{\xi_j\})$, and I_2 by I_{j+1}). The set $D = (F + \{\xi_1\} + \{\xi_2\} + \cdots + \{\xi_n\} + \cdots)$ then has the properties stated.

Definition. We say that the Hausdorff space $(X, \mathfrak{N})$ has the *property* P if, for each N, there is an N_1 such that $cN_1 \subseteq N$.

We remark that the property P does not imply that, for a given N and a given $x \in N$, there exists a $c[N_1(x)] \subseteq N$. Indeed this last condition is equivalent to regularity of the Hausdorff space. A regular Hausdorff space has the property P but there exist Hausdorff spaces having the property P which are irregular at each point.[2]

Corollary 4. *Let the Hausdorff space $(X, \mathfrak{N})$ have the property* P. *Let I_1 belong to $\mathfrak{J}$ and let T be any set of separated points in I_1. Then there exists an everywhere dense set, D, with $T \subseteq D \subseteq I_1$, such that f_D is continuous.*

Proof. The proof given for Corollary 3 is suitable with the modification that, in selecting a ξ_j from N^j ,we require that if N^j contains any points of T, then ξ_j shall be selected from among these. If no point of T is in N^j, then we find a $cN_j \subseteq N^j$ and select ξ_j from N_j. In this case $(T + \{\xi_1\} + \cdots + \{\xi_j\})$ remains a set of separated points and the construction can proceed.

[2] Examples of such spaces, as well as many helpful suggestions, were given to the writers by Professor H. P. Thielman.

REMARK. The set D in the last two corollaries is denumerable. That this result cannot be improved (assuming the continuum hypothesis) follows from the construction of Sierpinski and Zygmund[3] of a real function of a real variable, $f(x)$, such that, for any set, I, having the cardinality of the continuum, f_I is not continuous.

All the results of the present paper are of course applicable to an arbitrary operator on a separable Banach space. Somewhat related results are true for linear operators on an arbitrary Banach space[4] (the linearity of the operator making up for, in a sense, the lack of a denumerable basis).

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[3] *Sur une fonction qui est discontinue sur tout ensemble de puissance du continu*, Fund. Math. vol. 4 (1923) p. 316. Also W. Sierpinski, *Hypothese du continu*, Warsaw, 1934, p. 118. (Monografje Mathematyczne No. 4.)

[4] H. D. Block, *Linear transformations on or onto a Banach space*, Proceedings of the American Mathematical Society vol. 3 (1952) p. 126.