A SUFFICIENT CONDITION FOR POSITIVITY OF POLYNOMIAL FORMS[1]

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1. Introduction. In seeking conditions that certain algebraic expressions, primarily polynomials, in n indeterminates be positive, a sufficient condition was obtained when the values assigned to the indeterminates are positive numbers. The result was obtained through the use of Muirhead's theorem as given in §4. The methods used in obtaining the result have other possible applications, including an economic one, as given at the end of §2.

2. A distribution over a partially ordered system. In the plane, consider a finite set of points F_i $(i=1, \cdots, s)$. Partially order [**1**, §1] the set and represent the relation $F_j \leqq F_i$ by a vector from F_i to F_j. Denote the result by S. With each point F_i associate a real number f_i, to be called the *supply*, and thus get a *distribution* $(M|S)$, which is *excessive*, *balanced*, or *deficient* at F_i according as $0<f_i$, $0=f_i$, or $f_i<0$.

If $j\neq i$, define $F_j \prec F_i$ to mean $F_j \leqq F_i$ together with $f_j<0<f_i$. Also define $F_i \prec F_i$. Consider a pair of values i, j for which $F_j \prec F_i$. Select g_{ij} to satisfy

$$0 \leqq g_{ij} \leqq \min (f_i, -f_j).$$

Change the distribution $(M|S)$ into a distribution $(M'|S)$ by making the supply $f_i - g_{ij}$ at F_i, $f_j + g_{ij}$ at F_j, and f_k at F_k where $k\neq i, j$. The passage from $(M|S)$ to $(M'|S)$ will be called *supplying F_j with g_{ij} from F_i.*

If a distribution which is nowhere deficient, that is, has all supplies non-negative, can be found from $(M|S)$ by a finite number of supply operations, then $(M|S)$ will be called *adequate.*

If K is a subset of S, the notation

$$R(K) = \sum_K f_i$$

will be used for the sum of the supply of K.

A subset K of S will be called *complete* if $F_j \in K$ together with $F_j \leqq F_i$ implies $F_i \in K$. If K is complete, $R(K)$ is called a *reserve* for $(M|S)$. There is no essential restriction and considerable economy is

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involved if reserves are computed only for every complete subset whose graph is connected.

If K is any subset of S, the subset K' consisting of all points $F_j \in K$ together with all points F_i for which $F_j \leqq F_i$ will be called the *completion* of K. Then $K \subset K'$ and K' is complete.

THEOREM 2.1. *A distribution is adequate if and only if all its reserves are non-negative.*

The necessity of the condition follows from the observation that in the passage to a nowhere deficient distribution, all elements of a complete subset must be supplied from elements of that set.

To prove the sufficiency, consider the same points partially ordered with respect to $<$. This amounts to modifying the graph by erasing every vector which either starts at a point with nonpositive supply or ends at a point with non-negative supply. The resulting system S^* consists entirely of maximal points with $0 \leqq f_i$ and minimal points with $f_i \leqq 0$. There may, of course, be isolated points, which are both maximal and minimal.

Every operation of supply which can be executed in $(M|S)$ can also be executed in $(M|S^*)$ and conversely. Consequently $(M|S)$ is adequate if and only if $(M|S^*)$ is adequate.

If a subset is complete in S, it is also complete in S^*, but not conversely.

Suppose K is complete in S^* and consider its completion K' in S. If $K'-K$ is vacuous or consists only of points with nonpositive supply, then

$$0 \leqq R(K') = R(K) + R(K' - K) \leqq R(K).$$

If $K'-K$ contains points F_i with $0 < f_i$, then every point $F_j \in K$ for which $F_j \leqq F_i$ must have $0 \leqq f_j$. Removal of all such points F_j from K leaves a set $\overline{K}$ which is complete in S^* and

$$0 \leqq R(\overline{K}) < R(\overline{K}) + R(K - \overline{K}) = R(K).$$

Thus the reserves of S^* are non-negative when those of S are.

Adjust the notation so that $0 \leqq f_i$ or $f_i < 0$ according as $i < r$ or $r \leqq i$ and so that $F_r \leqq F_1$. Let non-negative g_{ij} be the amount to be supplied F_j from F_i for $1 \leqq i < r \leqq j \leqq s$. In this way an unknown g is associated with each vector in S^*.

Since no complete subset contains F_r without containing F_1, the conditions that $(M|S^*)$ have non-negative reserves fall into two categories, namely, those of the type

$$0 \leqq f_1 + A, \tag{2.1}$$

and those of the types

$$0 \leqq f_1 + f_r + B, \; 0 \leqq C, \tag{2.2}$$

where A, B, C are sums involving neither f_1 nor f_r.

Set

$$g_{ir} = \min (f_1, \; -f_r, \, f_1 + A). \tag{2.3}$$

Change the supply at F_1 to $f_1 - g_{1r}$, change that at F_r to $f_r + g_{1r}$, and erase the vector from F_1 to F_r to get a new distribution $(M^* | S^{**})$. The conditions that $(M^* | S^{**})$ have non-negative reserves are contained among relations of the types

$$0 \leqq (f_1 - g_{1r}) + A, \tag{2.4}$$

$$0 \leqq (f_1 - g_{1r}) + (f_r + g_{1r}) + B, \; 0 \leqq C, \tag{2.5}$$

which are implied by (2.1), (2.2), (2.3). Hence, $(M^* | S^{**})$ has non-negative reserves.

The operation can be repeated until all vectors have been erased. The final system then consists entirely of isolated points and the supply at each point is non-negative. This completes the proof.

For an economic application, suppose the countries F_i form an economic union. Read the symbol "$\leqq$" as "can import goods from," the ordering being assigned by the rules of the union. Let a_i be the number of tons of coal on hand, b_i the number of tons needed, and f_i the difference $a_i - b_i$. Theorem 2.1 tells whether the distribution of coal is adequate.

3. **Non-negative combinations of positive functions.** Consider the expression

$$F = f_1 F_1 + \cdots + f_s F_s, \tag{3.1}$$

where $f_1, \cdots, f_s$ are real numbers and $F_1, \cdots, F_s$ are functions whose values are positive and satisfy certain inequalities

$$F_j(x) \leqq F_i(x) \tag{3.2}$$

for all (x) in a fixed n-dimensional Euclidean domain V. Relations (3.2) partially order the F's. If the reserves of the distribution defined in §2 are non-negative, the function F is non-negative in V. If one reserve is positive, then F is positive.

Using the distribution $(M | S^*)$ and the notation of §2, we have

$$F = (f_1 - g_{1r})F_1 + (f_r + g_{1r})F_r + g_{1r}(F_1 - F_r) + \cdots .$$

Since $0 \leqq g_{1r}(F_1 - F_r)$,

$$(f_1 - g_{1r})F_1 + (f_r + g_{1r})F_r + \cdots \leqq F.$$

When all the supplies have been made non-negative, we have

$$f_1^* F_1 + \cdots + f_r^* F_r \leqq F,$$

so that F is non-negative, and if one f^* is positive, so is F.

4. **Symmetric forms.** Muirhead's theorem [3], also quoted in [2, §2.18] states that if all coordinates of (x) are positive, then

$$\omega(x;\, i) \leqq \omega(x;\, j)$$

if and only if $(i) \leqq (j)$, where

$$\omega(x;\, i) = \frac{1}{n!} \sum_{(x)} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

the summation being taken over all permutations of (x) for fixed (i). The notation $(i) \leqq (j)$ means

$$\sum_{p=0}^{n-1} i_{n-p} = \sum_{p=0}^{n-1} j_{n-p}, \qquad \sum_{p=0}^{k-1} i_{n-p} \leqq \sum_{p=0}^{k-1} j_{n-p} \qquad (k = 1, 2, \cdots, n-1),$$

where $i_1 \leqq i_2 \leqq \cdots \leqq i_n$, $j_1 \leqq j_2 \leqq \cdots \leqq j_n$. This is a partial ordering of the indices (i) and hence of the functions $\omega(x;\, i)$ of same degree.

Every symmetric form of degree p in n indeterminates can be written

$$\phi_n^p(x) = \sum_{i_1 + \cdots + i_n = p} f_{(i)} \omega(x;\, i). \tag{4.1}$$

The result of the preceding section gives sufficient conditions that ϕ be non-negative or positive.

If, as for a quadratic form, there are just two terms in ϕ and $(i) \leqq (j)$, then conditions are

$$0 \leqq f_{(j)}, \qquad 0 \leqq f_{(i)} + f_{(j)}.$$

The symmetric cubic form ϕ_n^3 has three terms corresponding to $(i_1) = (1, 1, 1, 0, \cdots, 0)$, $(i_2) = (2, 1, 0, \cdots, 0)$, $(i_3) = (3, 0, \cdots, 0)$ and can be written

$$\phi_n^3(x) = f_1\omega_1 + f_2\omega_2 + f_3\omega_3.$$

The sufficient conditions that it be non-negative are

$$0 \leqq f_1 + f_2 + f_3, \qquad 0 \leqq f_2 + f_3, \qquad 0 \leqq f_3. \tag{4.2}$$

These conditions, however, are not necessary. We have

$$\phi_n^3(x) = (f_1 + f_2 + f_3)\omega_1 + (f_2 + 2f_3)(\omega_2 - \omega_1) + f_3(\omega_1 - 2\omega_2 + \omega_3),$$
$$0 \leqq \omega_1 - 2\omega_2 + \omega_3$$

so that another set of sufficient conditions is

$$(4.3) \qquad 0 \leqq f_1 + f_2 + f_3, \qquad 0 \leqq f_2 + 2f_3, \qquad 0 \leqq f_3.$$

Conditions (4.2) imply (4.3), but not conversely.

References

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2. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1934.

3. R. F. Muirhead, *Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters*, Proceedings of the Edinburgh Mathematical Society vol. 21 (1903) pp. 144–157.

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