$$N > N_\epsilon \quad \text{and} \quad \lambda_\epsilon < \lambda < 1,$$

$$|D(x_0)| \leqq \sum_{j=0}^{\infty} I_j < \epsilon,$$

by the above inequalities and (4.1) of the paper cited in footnote 1. The same corollary as in that paper holds now for the functional of (1)—and for the same reasons as given in the proof for the bounded functional.

THE UNIVERSITY OF MINNESOTA

ESSENTIALLY ADMISSIBLE SEQUENCES

R. CREIGHTON BUCK

Let X be the set of all complex sequences $\alpha = \{a_n\}$ such that $\|\alpha\| = \sup_n |a_n|^{1/(n+1)} < \infty$. Under the usual operations, X is a complex vector space, and $\|\alpha+\beta\| \leqq \|\alpha\| + \|\beta\|$. However, $\|c\alpha\|$ is seldom $|c| \|\alpha\|$ and even though $c_n \to 0$, it is not in general true that $\|c_n\alpha\| \to 0$; for example, if $\alpha = \langle 1, 1, 1, \cdots \rangle$, then $\|c\alpha\|$ is c if $c \geqq 1$ and is 1 if $0 < c < 1$. Defining the distance between α and β as $\|\alpha - \beta\|$, X becomes a complete metric abelian group, but not a topological linear space. If with each α in X is associated the analytic function defined at the origin by $f(z) = \sum a_n z^n$, then this topology is that in which a sequence $\{f_n\}$ converges to the zero function only if on every bounded domain D, and for sufficiently large n, the functions f_n are all analytic on D and converge uniformly to zero; f_n converges to g if $f_n - g$ converges to zero. This topology is closely related to that introduced by Ganapathy Iyer into the vector space of all entire functions [4].

Given any $\alpha \in X$, there may be found an entire function $f(z)$ of order 1, finite type, and such that $f(iy) = O(\exp c|y|)$ for some $c \leqq \pi$, which interpolates to α in the sense that $f(n) = a_n$ for $n = 0, 1, 2, \cdots$ [1]. This is not the case if the condition $c < \pi$ is imposed. We have called a sequence α admissible in case such a more restricted function exists [2]. By a theorem of Carlson, such a function when it exists is unique [3]. A sequence α may fail to be admissible in an inessential way; for example, $\alpha = \langle 0, 0, 0, \cdots \rangle$ is admissible, but $\beta = \langle 1, 0, 0, \cdots \rangle$ is not. To allow for this, we shall now say that α

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is *essentially* admissible if there is a function $f(z)$ with $c<\pi$ such that $f(n)=a_n$ for all sufficiently large values of n. The set of all such α forms a subset of X which we denote by A. Because of the theorem of Carlson cited above, one expects A to be a "sparse" subset of X. The object of this note is to give this conjecture a precise form. We first observe that if α and β lie in A, so does $\alpha+\beta$, and $c\alpha$ for any complex c. Moreover, if α' is obtained from α by altering any finite set of its coordinates, α' is also in A. The study of A may be confined to that portion of it lying in the open unit sphere, since if $\|\alpha\|=R$ and $\alpha\in A$, the sequence $b_n=(1/2R)^{n+1}a_n$ is also in A, and $\|\beta\|=1/2$.

THEOREM 1. *A is of first category in X.*

In spite of this, it might be supposed that A is dense in X. This is not the case.

THEOREM 2. *A is not dense in X. In fact, there are open subsets of the unit sphere which contain no points of A.*

It will be seen that there are therefore spheres of arbitrarily large size (radius) which are free of points of A. We conjecture that A itself is nondense.

Before proving these, we introduce certain definitions. For any a and c, let $K(a, c)$ be the set of those entire functions $f(z)$ obeying $f(z)=O(1)$ $(\exp a|x|+c|y|)$, and let $A(a, c)$ be the corresponding subset of A, comprising those sequences α interpolated to by functions in $K(a, c)$. We have $A=\bigcup A(a, c)$ if the union is taken for all $0\leqq a<\infty$ and $0\leqq c<\pi$. Let A^* be the union of the closures of the sets $A(a, c)$. This of course may be only part of the closure of A.

Given any $\alpha\in X$, let $b_n=\Delta^n a_0=(-1)^n\sum C_{n,k}(-1)^k a_k$ and $g(z)=\sum b_n z^n$. We shall make use of a simple identity connecting $g(z)$ and the function $F(z)=\sum a_n z^n$. (See for example [5].)

$$(1)\quad (1+z)g(z)=F(z/(1+z)),\qquad (1-z)F(z)=g(z/(1-z)).$$

The basis for our proofs for the theorems stated above is the following characterization of the set A^*.

THEOREM 3. *$\alpha\in A^*$ if and only if $g(z)$ is regular at the origin and has an extension to a neighborhood of $-1\leqq x\leqq 0$ except possibly for an isolated singularity at -1.*

This depends upon the following characterization for the smaller class of admissible sequences [2]: *α is admissible if and only if $g(z)$ is regular on a neighborhood of the interval $-1\leqq x\leqq 0$.* This neighborhood contains an open set $\Omega(a, c)$ depending only on the growth constants a, c, of the entire function $f(z)$ which interpolates to α. Let α lie in the set $A(a, c)$. Choose an admissible $\beta=\{b_n\}$ such that $c_n=a_n-b_n=0$ for all $n>n_0$. We have

$$g(z) = \sum \Delta^n b_0 z^n + \sum \Delta^n c_0 z^n = g_1(z) + g_2(z).$$

Since β is admissible, g_1 is regular on $\Omega(a, c)$ containing $-1\leqq x\leqq 0$. Since $c_n=0$ for $n>n_0$, $F_2(z)=\sum c_n z^n$ is a polynomial, and, using (1), $g_2(z)$ has a pole at -1 as its only singularity. $g(z)$ is then regular in $\Omega(a, c)$ except for a pole at -1. Conversely, if this is true, $g(z)$ may be written as $g_1(z)+P(z)/(1+z)^m$ where P is a polynomial, and $a_n=b_n+c_n$ where $\beta=\{b_n\}$ is admissible, and $c_n=0$ for all large n. Let us now suppose that α is a limit point of $A(a, c)$. Given $\epsilon>0$, choose $\beta\in A(a, c)$ with $\|\alpha-\beta\|<\epsilon<1$, and let $\{c_n\}=\alpha-\beta$. Then $g(z) = \sum\Delta^n b_0 z+\sum\Delta^n c_0 z^n=g_1(z)+g_2(z)$. $g_1(z)$ is regular in $\Omega(a, c)$ except for a possible pole at -1. Since $|c_n|<\epsilon^{n+1}$ for all n, $\sum c_n z^n$ is regular for $|z|<1/\epsilon$ and, by (1), $g_2(z)$ is regular outside the disc $|z/(1+z)|\geqq 1/\epsilon$. $g(z)$ is then regular on the set obtained by deleting this disc from $\Omega(a, c)$. Letting ϵ decrease, $g(z)$ is regular on all of $\Omega(a, c)$ except possibly at -1. Conversely, let $g(z)$ be regular on a neighborhood Ω of $[-1, 0]$ except for an isolated singularity at -1. Write $g(z) =g_1(z)+g_2(z)=\sum\Delta^n b_0 z^n+\sum\Delta^n c_0 z^n$ where g_1 is regular in Ω, and g_2 has its only singularity at -1. Putting $\beta=\{b_n\}$, we see by the result cited above that β is admissible, and hence in $A(a, c)$ for a suitable choice of a and c. By (1), $\sum c_n z^n$ is entire, and $\lim |c_n|^{1/n}=0$. Setting $\gamma=\{c_n\}$, we have $\alpha=\beta+\gamma$, with $\beta\in A(a, c)$. It is not necessarily true that $\|\gamma\|$ is small. However, by a slight shift, we can show that α lies in the closure of $A(a, c)$. For any N, set

$$c_n' = \begin{cases} 0, & n \leqq N, \\ c_n, & n > N, \end{cases} \qquad b_n' = \begin{cases} b_n + c_n, & n \leqq N, \\ b_n, & n > N. \end{cases}$$

Then, $\alpha=\beta'+\gamma'$, and $\|\alpha-\beta'\|=\|\gamma'\|=\sup_{n\geqq N} |c_n|^{1/(n+1)}$, which may be made arbitrarily small by increasing N. Since β' agrees with β at all but a finite number of coordinates, β' lies in $A(a, c)$.

We next proceed to the proof of Theorem 1 and Theorem 2.

Proof of Theorem 1. Since $A=\bigcup_{n,m} A(n, \pi-1/m)$, we shall show that A is of first category if we show that each $A(a, c)$ is nondense. Given $\beta\in A(a, c)$ and $\epsilon>0$, we shall produce α such that $\|\beta-\alpha\|<\epsilon$, but such that α is not in the closure of $A(a, c)$. Let $\gamma=\{c_n\}$ where c_n is 0 if n is a square, and ϵ^{n+1} otherwise, and set $\alpha=\beta+\gamma$. Clearly, $\|\gamma\|=\epsilon$. The function $\sum c_n z^n=\sum\epsilon^{k^2+1}z^{k^2}$ has the circle $|z|=1/\epsilon$ as a cut. By (1), $g_2(z)=\sum\Delta^n c_0 z^n$ is regular for $\epsilon|z|<|1+z|$, and the boundary of this is a cut. Since β is in $A(a, c)$, $g_1(z)=\sum\Delta^n b_0 z^n$ is regular on a neighborhood of $-1 \leqq x \leqq 0$, and $g(z) = g_1(z) + g_2(z)$ does not have the type of behavior which permits α to be in the set A^*.

PROOF OF THEOREM 2. We must show that there are open sets in X which contain no points of A. For this we use a special theorem concerning oscillating sequences: *if C_1 and C_2 are two disjoint convex sets and if the terms of the sequence α alternate between these sets, α is not admissible* [1]. Let $c > 1$ and consider the special sequence $\beta = \{b_n\}$ where $b_n = c(-1)^n$. If, now, $\|\alpha - \beta\| \leqq 1$, then $|a_n - c(-1)^n| \leqq 1$ for all n, so that a_n alternates between the circles of unit radius with center at c and $-c$. Thus, the sphere, center β and radius 1, is disjoint from A. More generally, if $b_n = (-1)^n R^{n+1}$ then the open sphere, center β and radius R, is disjoint from A. It should be noted that $\|\beta\| = R$, so that these A-free spheres can be found at any distance from the origin.

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UNIVERSITY OF WISCONSIN