MINIMIZING OPERATORS ON SUBREGIONS[1]

LEO SARIO

For constructing harmonic functions with a prescribed local behavior, an operator method on arbitrary Riemann surfaces was recently introduced by the author [1]. We showed the existence of a normal linear operator minimizing the Dirichlet integral and referred to other operators to be given later. In the present paper, a general class of minimizing operators will be introduced, including the above operator as a special case. In the existence proof, use will be made of the extremal method presented in [2].

Let R be an arbitrary Riemann surface and G a subregion, compact or not, of finite or infinite genus, relatively bounded by a finite set α of closed analytic Jordan curves. Let v be a real single-valued function on α, harmonic in an open set containing α. A normal linear operator L in G is defined [1] as follows. With every v on α is associated, by L, a unique single-valued harmonic function Lv on G which satisfies the following conditions:

$$Lv = v \text{ on } \alpha, \tag{1}$$

$$\min_\alpha v \leqq Lv \leqq \max_\alpha v \text{ on } G, \tag{2}$$

$$\int_\alpha d\overline{Lv} = 0, \tag{3}$$

$$L(c_1v_1 + c_2v_2) = c_1Lv_1 + c_2Lv_2. \tag{4}$$

Here $\overline{Lv}$ is the harmonic conjugate function of Lv.

Denote by $\{u\}$ the class of single-valued harmonic functions u in G with

$$u = v \text{ on } \alpha, \qquad \int_\alpha d\bar{u} = 0. \tag{5}$$

Let β be the ideal boundary of G, that is, the common part of the boundaries of R and G. If G is noncompact (β is not empty), we form an exhaustion $\{G_n\}$ of G by domains G_n, bounded by α and a finite set β_n of closed analytic Jordan curves. The boundary integral $\int_\beta u d\bar{u}$ is defined as the limit of integrals taken along the curves β_n.

Received by the editors June 30, 1952.

[1] This investigation was carried out under the partial sponsorship of the Office of Ordnance Research, Contract DA-034-ORD-639 RD.

These integrals increase monotonically with n, for the Dirichlet integral of u over $G_{n+1}-G_n$ is non-negative. If G is compact (β is empty), the boundary integral is understood to vanish for all u. Let λ be a real parameter ranging in the interval $(-1, 1)$.

THEOREM 1. *There is a uniquely determined function u_λ in G which minimizes the value of the functional*

$$m_\lambda(u) = \int_\beta u d\bar{u} + \lambda \int_\alpha u d\bar{u} \tag{6}$$

among all functions of the class $\{u\}$. The function u_λ is associated with v by a normal linear operator L_λ,

$$u_\lambda = L_\lambda v. \tag{7}$$

PROOF. If G is compact, $\{u\}$ reduces to one single function and the theorem is trivial. In the sequel we assume that G is not compact. Suppose first that β consists of a finite number of closed analytic Jordan curves. Let u_1 and u_{-1} be the functions of class $\{u\}$ determined by

$$u_1 = k = \text{const. on } \beta, \tag{8}$$

$$\partial u_{-1}/\partial n = 0 \text{ on } \beta \tag{9}$$

where $\partial/\partial n$ is the normal derivative. Write

$$u_\lambda = \frac{1+\lambda}{2} u_1 + \frac{1-\lambda}{2} u_{-1}, \tag{10}$$

and set $u - u_\lambda = h$. By $h = 0$ on α, we have

$$\begin{aligned} m_\lambda(u) = \int_\beta u_\lambda d\bar{u}_\lambda + \lambda \int_\alpha u_\lambda d\bar{u}_\lambda + \int_{\beta-\alpha} h d\bar{h} \\ + \int_\beta u_\lambda d\bar{h} + \lambda \int_\alpha u_\lambda d\bar{h} + \int_{\beta-\alpha} h d\bar{u}_\lambda. \end{aligned}$$

In view of the Green's formula

$$\int_{\beta-\alpha} h d\bar{u}_\lambda = \int_{\beta-\alpha} u_\lambda d\bar{h},$$

the sum of the three latter integrals may be written

$$2 \int_\beta u_\lambda d\bar{h} + (\lambda - 1) \int_\alpha u_\lambda d\bar{h}.$$

Substituting (10) in this and making use of (5), this reduces further to

$$(1-\lambda)\int_{\beta-\alpha} u_{-1}d\bar{h} = (1-\lambda)\int_{\beta-\alpha} hd\bar{u}_{-1} = 0.$$

Consequently,

$$m_\lambda(u) = m_\lambda(u_\lambda) + D(u - u_\lambda), \tag{11}$$

which shows that $m_\lambda(u)$ is minimized by u_λ. By (8) and (9), the functions u_1 and u_{-1} are associated with v by a normal linear operator. The same is, therefore, true for u_λ. This proves the theorem for the special β under consideration.

Now let β be arbitrary. Denote by $u_{\lambda n}$ the harmonic function in G_n which minimizes the value of

$$m_{\lambda n}(u) = \int_{\beta_n} ud\bar{u} + \lambda\int_\alpha ud\bar{u}$$

among functions of the class $\{u\}$ in G_n. By (2), the functions $u_{\lambda n}$ are uniformly bounded, and a subsequence, say again $\{u_{\lambda n}\}$, converges uniformly in every closed subdomain of G towards a harmonic function u_λ on G with $u_\lambda = v$ on α. In view of the harmonic boundary values and Schwarz's reflexion principle, the convergence is uniform even in a domain slightly extended across α. This implies that grad $u_{\lambda n}$ converges uniformly on α.

Since $\int_{\beta_n} u_{\lambda m} d\bar{u}_{\lambda m}$ increases with $n(\leqq m)$, it follows from the minimum property of $u_{\lambda n}$ that

$$m_{\lambda n}(u_{\lambda n}) \leqq m_{\lambda(n+1)}(u_{\lambda(n+1)}).$$

Similarly, for u in G,

$$m_{\lambda n}(u_{\lambda n}) \leqq m_\lambda(u).$$

As this holds for every n and every u in G, we have

$$\lim_{n\to\infty} m_{\lambda n}(u_{\lambda n}) \leqq \inf m_\lambda(u) \leqq m_\lambda(u_\lambda).$$

Since, on the other hand,

$$m_\lambda(u_\lambda) = \lim_{n\to\infty} m_{\lambda n}(u_\lambda) = \lim_{n\to\infty}\lim_{m\to\infty} m_{\lambda n}(u_{\lambda m}) \leqq \lim_{m\to\infty} m_{\lambda m}(u_{\lambda m}),$$

it follows that

$$m_\lambda(u_\lambda) = \min m_\lambda(u) = \lim_{n\to\infty} m_{\lambda n}(u_{\lambda n}). \tag{12}$$

This minimum property implies that, for a real ϵ,

$$m_\lambda(u_\lambda + \epsilon h) = m_\lambda(u_\lambda) + \epsilon\left[\int_\beta (u_\lambda d\bar{h} + h d\bar{u}_\lambda) + \lambda \int_\alpha (u_\lambda d\bar{h} + h d\bar{u}_\lambda)\right] + \epsilon^2 D(h).$$

The expression in brackets vanishes, since otherwise, for sufficiently small $|\epsilon|$, the deviation of $m_\lambda(u_\lambda+\epsilon h)$ from $m_\lambda(u_\lambda)$ would change its sign with ϵ, contrary to the minimum property of u_λ. For $\epsilon=1$, it follows that

$$m_\lambda(u) = m_\lambda(u_\lambda) + D(u - u_\lambda).$$

This guarantees the uniqueness of u_λ. In fact, let u' and u'' be two minimizing functions. Then

$$m_\lambda(u'') = m_\lambda(u') = m_\lambda(u'') + D(u' - u'')$$

which implies $u'-u''=\text{const.}=0$. In particular, the sequence $u_{\lambda n}$, not only a subsequence, converges.

Since $u_{\lambda n}=L_{\lambda n}v$ satisfies the conditions (1)–(4) in G_n, it follows from the uniform convergence that L_λ, defined by

$$u_\lambda = L_\lambda v,$$

is a normal linear operator for G. This completes the proof of Theorem 1.

We consider now the subclass $\{u^0\}$ of $\{u\}$, defined by the restriction $\int d\bar{u}^0=0$ along all dividing cycles.

THEOREM 2. *There is a normal linear operator $\overset{0}{L_\lambda}$ associating with v on α a unique harmonic function*

(13) $$\overset{0}{u_\lambda} = \overset{0}{L_\lambda} v$$

on G which minimizes the value

$$m_\lambda(u^0) = \int_\beta u^0 d\bar{u}^0 + \lambda \int_\alpha u^0 d\bar{u}^0$$

among all functions of the class $\{u^0\}$.

PROOF. In the proof of Theorem 1, replace u_1 by $\overset{0}{u_1}\in\{u^0\}$, defined by

(14) $$\overset{0}{u_1} = k_{ni} = \text{const. on } \beta_{ni},$$

where β_{ni} are the closed curves constituting β_n. Write $\overset{0}{u}_{-1}\equiv u_{-1}$ and replace u_λ by $\overset{0}{u_\lambda}$, respectively. Then nothing in the previous proof will be changed if the exhaustion $\{G_n\}$ is (as is always possible) chosen so that each β_{ni} is a dividing cycle.

We now apply the operators introduced above to existence problems on the Riemann surface R, on which the subregion G was considered. In $R-\alpha$, let s be a single-valued real function, harmonic near α, both branches of which can be continued harmonically across α. Let L be a normal linear operator in $R-\alpha$. The following theorem was proved in [1]. If $\int d\bar{s}$ vanishes, when extended along both edges of α for respective branches of s, then, and only then, there exists on the whole surface R a function p, harmonic on α and such that $p-s=L(p-s)$ in each of the disjoint regions constituting $R-\alpha$. For $L=L_\lambda$ (or L_λ^0) this gives, in particular:

THEOREM 3. *On an arbitrary Riemann surface R, let D be a compact region, bounded by a finite set α of closed analytic Jordan curves. In D, let s be a single-valued real function, harmonic on α. The condition*

$$\int_\alpha d\bar{s} = 0 \tag{15}$$

is necessary and sufficient for the existence of a single-valued function p_λ (or p_λ^0) on R such that

1. *$p_\lambda - s$ is harmonic on $\bar{D}$,*
2. *p_λ is harmonic on $R-D$,*
3. *the value of the functional*

$$m_\lambda(u) = \int_\beta u d\bar{u} + \lambda \int_\alpha u d\bar{u} \tag{16}$$

is minimized by $u=p_\lambda$ among all functions of the class $\{u\}$ (or $\{u^0\}$) in $R-D$ with the boundary values p_λ on α.

The proof is furnished by the theorem quoted above, selecting $s\equiv 0$ in $R-\bar{D}$.

Note that, for $\lambda=-1$, the operator L_λ is the special operator introduced in [1] (denoted there by L_0) which minimizes the Dirichlet integral. For $\lambda=1$, L_λ minimizes $\int_{\beta-\alpha} u d\bar{u}$, furnishing the function of Lemma 1 in [4]. For $\lambda=0$, $\int_\beta u d\bar{u}$ is minimized by L_λ, the mean of the two above operators. Necessary and sufficient conditions, given in [1] for the existence of certain harmonic and analytic functions, are valid in terms of any of the operators L_λ.

The functions p_{-1}^0 and p_1^0, corresponding to the operators L_{-1}^0 and L_1^0 and to $s=\text{Re }(1/z)$, are the real parts of functions mapping a planar surface onto the horizontal or vertical slit domains, respectively. The functions p_λ^0 have application to related mapping problems.

A survey of the linear operator method and the extremal method, to which this investigation is related, was given in [3].

References

1. Leo Sario, *A linear operator method on arbitrary Riemann surfaces*, Trans. Amer. Math. Soc. vol. 72 (1952) pp. 281–295.

2. ———, *An extremal method on arbitrary Riemann surfaces*, Trans. Amer. Math. Soc. vol. 73 (1952) pp. 459–470.

3. ———, *Construction of functions with prescribed properties on Riemann surfaces*, Conference on Riemann Surfaces in Commemoration of the Hundredth Anniversary of Riemann's Inaugural Dissertation, Princeton, 1951, Annals of Mathematics Studies (to appear).

4. ———, *Alternating method on arbitrary Riemann surfaces*, Pacific Journal of Mathematics (to appear).

Stanford University