DEFINITIONS OF GROUP INVOLVING QUASI-INVERSE ELEMENTS

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1. **Introduction.** In this paper we present definitions for groups in which the notions of identity and inverse elements employed in earlier definitions (see Moore [1, pp. 485–487, 491–492] and Dickson [2, pp. 198–201])[1] are replaced by the notion of *quasi-inverse* elements. By a *right* (*left*) *quasi-inverse* of an element a in a groupoid Γ with operation $\circ$, we shall mean an element $a'('a)$ in the groupoid such that $(b \circ a) \circ a' = b('a \circ (a \circ b) = b)$ for every b in Γ. It should be observed that since an inverse element is not describable without the notion of an identity element, the independence of the identity postulate from the inverse postulate is only vacuously verified. This not being the case with quasi-inverse elements, in our definitions the vacuous satisfaction of postulates is met with only in the case of the independence system for the closure postulate. We give three independent sets of postulates for groups, and four independent sets for Abelian groups. In the formulation of the latter sets, which are rather simple, we introduce modified versions of the associative law and of the quasi-inverse postulates.

2. **List of postulates.** In order to avoid repetition we give here a list of eleven conditions on an undefined class K and a binary operation $\circ$ from which we select the various sets of postulates so that the system $(K, \circ)$ shall be a group or an Abelian group respectively. In II, II_1, II_2, supply the clause: *whenever a, b, c, and the indicated combinations are in K*; in III_1 and III_4, supply the clause: *whenever b and $a \circ b$ are in K*; and in III_2 and III_3, supply the clause: *whenever b and $b \circ a$ are in K*. The list follows.

I. *$a \circ b$ is in K whenever a, b are in K.*

II. $(a \circ b) \circ c = a \circ (b \circ c)$.

II_1. $(a \circ b) \circ c = (b \circ c) \circ a$.

II_2. $a \circ (b \circ c) = b \circ (c \circ a)$.

III_1. *Corresponding to every a in K, there exists an element $'a$ such that $'a \circ (a \circ b) = b$.*

III_2. *Corresponding to every a in K, there exists an element a' such that $(b \circ a) \circ a' = b$.*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

III_3. *Corresponding to every a in K, there exists an element a' such that* $a' \circ (b \circ a) = b$.

III_4. *Corresponding to every a in K, there exists an element a' such that* $(a \circ b) \circ a' = b$.

C_1. $a \circ b = a \circ c$ *implies* $b = c$.

C_2. $b \circ a = c \circ a$ *implies* $b = c$.

N. *K has at least two distinct elements.*

3. **Postulate-sets for groups.** From the foregoing list of conditions, we form the following three postulate-sets each of which constitutes a definition of a nontrivial group. It is clear that in each set we may omit postulate N should the group under consideration be an unrestricted one. The sets follow.

Set G_1: I, II, III_1, III_2, N.

Set G_2: I, II, III_1, C_2, N.

Set G_3: I, II, III_2, C_1, N.

4. **Theorems from set** G_1. The following theorems T.4.1–T.4.5 derivable from the postulates of set G_1 establish the adequacy of these postulates for a (nontrivial) group.

T.4.1. $$'a = a'.$$

PROOF. $'a = ('a \circ a) \circ a' = 'a \circ (a \circ a') = a'$, by III_2, II, III_1.

T.4.2. $$a \circ a' = a' \circ a.$$

PROOF. $a \circ a' = ['a \circ (a \circ a)] \circ a' = 'a \circ [(a \circ a) \circ a'] = 'a \circ a = a' \circ a$, by III_1, II, III_2, T.4.1.

T.4.3. $$c \circ c' = a \circ a' \textit{ for every } c \textit{ in } K.$$

PROOF. $c \circ c' = 'a \circ [a \circ (c \circ c')] = 'a \circ [(a \circ c) \circ c'] = 'a \circ a = a' \circ a = a \circ a'$, by III_1, II, III_2, T.4.1, T.4.2.

DEFINITION D. $a \circ a' = u$.

T.4.4. $$a \circ u = a.$$

PROOF. $a \circ u = a \circ (a \circ a') = (a \circ a) \circ a' = a$, by D, II, III_2.

In view of I, II, T.4.4, D, and N, we have the following theorem.

T.4.5. *The system $(K, \circ)$ is a (nontrivial) group.*

5. **Theorems from sets** G_2 **and** G_3. The following sequence of theorems T.1–T.4, derivable from the postulates of set G_2, will serve to establish the adequacy of these postulates for a (nontrivial) group. The same theorem-sequence will hold for set G_3, except that in every

proposition where the mark $'$ occurs, its position should be at the right (instead of the left) of each element affected by it. The proofs can be readily supplied by the reader. The theorem-sequence follows.

T.1. $a \circ {}'a = {}'a \circ a.$

T.2. $c \circ {}'c = a \circ {}'a$ *for every* c *in* K.

DEFINITION D. $a \circ {}'a = u$.

T.3. $a \circ u = a.$

T.4. *The system* $(K, \circ)$ *is a (nontrivial) group.*

6. **Postulate-sets for Abelian groups.** From the list of conditions given in §2, we form the following four postulate-sets each of which constitutes a definition of a nontrivial Abelian group. It is clear that in each set we may omit N should the Abelian group under consideration be an unrestricted one. The sets follow.

Set H_1: I, II, III_4, N.
Set H_2: I, II, III_3, N.
Set H_3: I, II_1, III_2, N.
Set H_4: I, II_2, III_1, N.

7. Theorem-sequences S_i $(i=1, 2, 3, 4)$ made up of propositions taken from the following list are given below, each S_i serving to establish the adequacy of the postulates of the similarly numbered H_i for a (nontrivial) Abelian group. In S_4 the position of the mark $'$, in every proposition where it occurs, should be at the left (instead of the right) of each element affected by it. The list of propositions follows:

$T.1_1$. $a \circ b = a \circ c$ *implies* $b = c$.

$T.1_2$. $b \circ a = c \circ a$ *implies* $b = c$.

T.2. $a \circ a' = a' \circ a.$

T.3. $c \circ c' = a \circ a'$ *for every* c *in* K.

DEFINITION D. $a \circ a' = u$.

$T.4_1$. $u \circ a = a$ *for every* a *in* K.

$T.4_2$. $a \circ u = a$ *for every* a *in* K.

T.5. $a \circ b = b \circ a.$

T.6. $(a \circ b) \circ c = a \circ (b \circ c).$

T.7. *The system* $(K, \circ)$ *is a (nontrivial) Abelian group.*

The theorem-sequences follow.
S_1: T.1_1, T.2, T.3, D, T.4_2, T.5, T.7.
S_2: T.1_2, T.2, T.3, D, T.4_1, T.4_2, T.5, T.7.
S_3: T.2, T.3, D, T.4_1, T.4_2, T.5, T.6, T.7.
S_4: T.2, T.3, D, T.4_2, T.4_1, T.5, T.6, T.7.

8. **Proofs of the theorems.** In this section we give the proofs of the theorems of S_1. The proofs of those of S_2–S_4 are left to the reader.

PROOF OF T.1_1. $b=(a \circ b) \circ a'=(a \circ c) \circ a'=c$, by III_4, hypothesis, III_4.

PROOF OF T.2. $(a \circ a') \circ (a \circ a')=a \circ [a' \circ (a \circ a')]=a \circ [(a' \circ a) \circ a']=[a \circ (a' \circ a)] \circ a'=a' \circ a=[(a \circ a') \circ a'] \circ a=(a \circ a') \circ (a' \circ a)$, by II, II, II, III_4, III_4, II. Hence, $a \circ a'=a' \circ a$, by T.1_1.

PROOF OF T.3. $c \circ c'=[(a \circ c) \circ a'] \circ c'=[a \circ (c \circ a')] \circ c' =a \circ [(c \circ a') \circ c']=a \circ a'$, by III_4, II, II, III_4.

PROOF of T.4_2. $a \circ u=a \circ (a \circ a')=(a \circ a) \circ a'=a$, by D, II, III_4.

PROOF OF T.5. $a \circ b=[(b \circ a) \circ b'] \circ b=(b \circ a) \circ (b' \circ b)=(b \circ a) \circ (b \circ b')=(b \circ a) \circ u=b \circ a$, by III_4, II, T.2, D, T.4_2.

PROOF of T.7. This follows from I, II, T.4_2, D, T.5, N.

9. **Independence of the postulates.** The independence of the postulates of G_1–G_3 and H_1–H_4 is established by the systems given in the table below, each system being in a row with the postulate-set in question and each of its component independence examples being in a column with the contradicted postulate. Each system is taken from the following list of independence examples. In every example, except the last, $a \circ b$ is given by a multiplication-table. In example (1) the blank indicates that the result of the corresponding combination is not in K. The list of independence examples follows.

(1) $K = 1, 2$;

	1	2
1	–	2
2	2	1

(2) $K = 1, 2, 3$;

	1	2	3
1	2	1	3
2	1	3	2
3	3	2	1

(3) $K = 1, 2$;

	1	2
1	1	1
2	2	2

(4) $K = 1, 2$;

	1	2
1	1	2
2	1	2

(5) $K = 1, 2$;

	1	2
1	1	1
2	1	1

(6) $K = 1; a \circ b = 1$

TABLE OF INDEPENDENCE SYSTEMS

SET	I	II	II_1	II_2	III_1	III_2	III_3	III_4	C_1	C_2	N
G_1	(1)	(2)			(3)	(4)					(6)
G_2	(1)	(2)			(3)					(4)	(6)
G_3	(1)	(2)				(4)			(3)		(6)
H_1	(1)	(2)						(5)			(6)
H_2	(1)	(2)					(5)				(6)
H_3	(1)		(2)			(5)					(6)
H_4	(1)			(2)	(5)						(6)

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