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TWO NOTES ON RECURSIVELY ENUMERABLE SETS

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Introduction. These notes are based on E. L. Post's paper *Recursively enumerable sets of positive integers and their decision problems*[1] to which we shall refer as RES. The reader is assumed to be familiar with §§1–5 and 9 of this paper. In the first note we shall discuss some algebraic properties of simple and hypersimple sets. In the second note we shall prove the existence of a recursively enumerable set which is neither recursive nor creative nor simple and discuss its degree of unsolvability relative to one-one reducibility and relative to many-one reducibility.

Notations and terminology. A collection of non-negative integers is called a *set*, a collection of sets is called a *class*. An empty collection is considered as a special case of a finite collection. Non-negative integers and functions are denoted by small Latin letters, sets by small Greek letters, and classes by capital Latin letters. The Boolean operations are denoted by "$+$" for addition, "$\times$," "$\cdot$" or juxtaposition for multiplication, "$'$" for complementation and "$\subset$" for inclusion. Proper inclusion between classes is denoted by "$\subset_+$."

$\epsilon =_{df}$ the set of all non-negative integers.
$o =_{df}$ the empty set.
$\kappa =_{df}$ the complete set defined on p. 295 of RES.
$\zeta =_{df}$ the simple set defined on p. 298 of RES.
$P =_{df}$ the class of all sets whose complement is finite.
$Q =_{df}$ the class of all finite sets.
$E =_{df}$ the class of all recursive sets.
$D =_{df} E-(P+Q)$.

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[1] Bull. Amer. Math. Soc. vol. 50 (1944) pp. 284–316.

$F =_{df}$ the class of all recursively enumerable (r.e.) sets.
$Z =_{df}$ the class of all simple sets.
$Z_0 =_{df}$ the class of all hypersimple sets.

NOTE 1. SOME PROPERTIES OF SIMPLE SETS

1. **Preliminaries.** A set is called *immune* if it is infinite, but has no infinite r.e. subset; a set α is called *simple* if α is r.e. and α' immune. Clearly $Z \subset_+ F - E$. The function $f(n)$ is called a *recursive permutation* if it is a recursive function which maps ϵ 1-1 on itself. The sets α and β are called *isomorphic* (notation: $\alpha \cong \beta$) if there exists a recursive permutation which maps α on β. The class S is called *recursively closed* if it contains with any set α also all sets which are isomorphic with α. The classes P, Q, E, D, F, Z, Z_0 are obviously recursively closed. Suppose L is a class of sets which is a lattice relative to $+$ and $\times$. The subclass S of L is called a *dual ideal* in L if: (1) S is closed under $\times$, (2) if $\alpha \in S$ and $\beta \in L$, then $\alpha + \beta \in S$. It is easily verified that the second condition may be replaced by (2*) if $\alpha \in S$, $\beta \in L$, and $\alpha \subset \beta$, then $\beta \in S$. Observe that E is a Boolean algebra, while F is a distributive lattice with a null element (namely o) and a one element (namely ϵ).

THEOREM 1.1. *Any two sets in D are isomorphic.*

PROOF. If $\gamma, \delta \in D$ there exist 1-1 recursive functions $c(n)$, $c'(n)$, $d(n)$, $d'(n)$ which range over γ, γ', δ, δ' respectively. Let $f(c(n)) =_{df} d(n)$ and $f(c'(n)) =_{df} d'(n)$, then it is easily verified that $f(n)$ is a recursive permutation which maps γ on δ.

THEOREM 1.2. *If $\sigma \in F - Q$ and $\delta \in D$, we can find a set τ such that $\delta \subset \tau$ and $\tau \cong \sigma$.*

PROOF. We can effectively find a set $\gamma \in D$ which is a subset of σ. By the preceding theorem there exists a recursive permutation which maps γ on δ, say $f(n)$. Let $\tau = f(\sigma)$, then $\tau \cong \sigma$; moreover $\gamma \subset \sigma$ implies $f(\gamma) \subset f(\sigma)$, i.e. $\delta \subset \tau$.

THEOREM 1.3. *The r.e. set σ is simple if and only if $\sigma' \notin Q$ and $\sigma \cdot \alpha \notin Q$ for every $\alpha \in F - Q$.*

PROOF. We can restrict our attention to the "only if" part, since the "if" part is obvious. If σ is simple, then σ' is infinite, because σ' is immune. Moreover $\sigma \cdot \alpha = o$ for $\alpha \in F - Q$ is impossible, since it would imply $\sigma' \supset \alpha$, while σ' is immune. But if σ had only finitely many elements in common with the set $\alpha \in F - Q$, σ would have no element in common with the set $\alpha - \sigma \cdot \alpha \in F - Q$. Thus $\sigma \cdot \alpha \notin Q$ for every

$\alpha \in F-Q$.

The infinite sequence $\{\alpha_n\}$ of nonempty, finite sets is called *strictly r.e.* if there exist recursive functions $a(m, n)$ and $b(n)$ such that for every n, $\alpha_n = \{a(n, 0), \cdots, a(n, b(n))\}$. A strictly r.e. infinite sequence $\{\alpha_n\}$ of nonempty finite sets is called an *array*, the elements α_n of $\{\alpha_n\}$ are called the *rows* of the array. The array $\{\alpha_n\}$ is called *discrete* if α_m and α_n are disjoint for $m \neq n$. We say that the set α includes the ith row of the array $\{\beta_n\}$ if $\alpha \supset \beta_i$. The set α is called *hypersimple* if α is r.e., α' infinite, and α includes at least one row of every discrete array. Every hypersimple set is simple, since the recursive function $b(n)$ mentioned above may be identically 0. Post proved the existence of a hypersimple set [RES pp. 305–308] and the existence of a simple set which is not hypersimple [RES p. 298]. Hence $Z_0 \subset_+ Z$.

THEOREM 1.4. *The r.e. set σ is hypersimple if and only if $\sigma' \notin Q$ and σ includes infinitely many rows of every discrete array.*

PROOF. We can restrict our attention to the "only if" part, the "if" part being obvious. Let σ be hypersimple; then $\sigma' \notin Q$ since σ is simple. Suppose σ included only finitely many rows of the discrete array $\{\alpha_n\}$. Let r be the greatest number n such that $\sigma \supset \alpha_n$. Then σ would include no row of the discrete array $\{\alpha_{(r+1)+n}\}$; this would contradict the fact that σ is hypersimple.

2. The main result.

We can now prove some algebraic properties of simple and hypersimple sets.

THEOREM 1.5. (1) *The product of two simple sets is simple.*

(2) *The sum of two simple sets is either simple or belongs to P.*

(3) *There exist two simple sets whose sum equals ϵ.*

(4) *$Z+P$ is a dual ideal in the lattice F.*

PROOF. (1) Let $\alpha, \beta \in Z$. By Theorem 1.3 it is sufficient to prove that $(\alpha\beta)' \notin Q$ and that $\alpha\beta \cdot \gamma \notin Q$ for every $\gamma \in F-Q$. Clearly $(\alpha\beta)'$ is infinite, since α' is infinite. Suppose $\gamma \in F-Q$; then $\beta\gamma \in F-Q$ because $\beta \in Z$, and $\alpha\beta \cdot \gamma = \alpha \cdot \beta\gamma \in F-Q$ because $\alpha \in Z$ and $\beta\gamma \in F-Q$.

(2) Let $\alpha, \beta \in Z$. Either $\alpha+\beta \in P$ or $\alpha+\beta \notin P$. In the latter case $\alpha+\beta \in Z$, since $\alpha \subset \alpha+\beta$ and $\alpha \in Z$.

(3) Let $\delta \in D$, $\sigma \in Z$. Then $\delta' \in D$ and by Theorem 1.2 there exist sets α and β such that $\delta \subset \alpha$, $\delta' \subset \beta$, $\alpha \leqq \sigma$, $\beta \leqq \sigma$. Then α and β are simple, since Z is recursively closed; moreover $\delta+\delta' \subset \alpha+\beta$, hence $\alpha+\beta=\epsilon$.

(4) The product of two sets in Z is in Z, the product of two sets in P is in P, and the product of a set in Z and a set in P is in Z. Thus

$Z+P$ is closed under the product operation. The proof of part (2) remains valid if we replace the assumptions $\alpha, \beta \in Z$ by $\alpha \in Z, \beta \in F$. Let $\alpha \in Z+P$ and $\beta \in F$; then either $\alpha \in Z$ hence $\alpha+\beta \in Z+P$, or $\alpha \in P$ hence $\alpha+\beta \in P$. Thus $\alpha+\beta \in Z+P$. This completes the proof.

THEOREM 1.6. (1) *The product of two hypersimple sets is hypersimple.*

(2) *The sum of two hypersimple sets is either hypersimple or belongs to P.*

(3) *There exist two hypersimple sets whose sum equals ϵ.*

(4) *Z_0+P is a dual ideal in the lattice F.*

PROOF. (1) Let $\alpha, \beta \in Z_0$ and let $\{\gamma_n\}$ be a discrete array. Suppose ρ is the set of all non-negative integers n such that $\beta \supset \gamma_n$, then ρ is infinite by Theorem 1.4. Let Γ_i be the act of comparing the first i elements of β with the first i rows of $\{\gamma_n\}$, then we can effectively generate ρ by performing the acts $\Gamma_0, \Gamma_1, \cdots$. Thus ρ is r.e.; suppose $r(n)$ is a 1-1 recursive function ranging over ρ and suppose $\delta_n =_{df} \gamma_{r(n)}$. Then $\{\delta_n\}$ is a discrete array which is a subarray of $\{\gamma_n\}$. Since $\beta \supset \delta_n$ for all values of n and $\alpha \supset \delta_n$ for infinitely many values of n, it follows that $\alpha\beta \supset \gamma_n$ for infinitely many values of n. We conclude $\alpha\beta \in Z_0+P$. Clearly $\alpha\beta \notin P$ because $\alpha \notin P$. Thus $\alpha\beta \in Z_0$.

(2) Let $\alpha, \beta \in Z_0$. Either $\alpha+\beta \in P$ or $\alpha+\beta \notin P$. In the latter case $\alpha+\beta \in Z_0$, since $\alpha \subset \alpha+\beta$ and $\alpha \in Z_0$.

(3) Using the fact that Z_0 is recursively closed we can prove this part similarly to the third part of Theorem 1.5.

(4) The proof of part (2) remains valid if we replace the assumptions $\alpha, \beta \in Z_0$ by $\alpha \in Z_0, \beta \in F$. We can now prove this part in the same way as the fourth part of Theorem 1.5.

NOTE 2. A MESOIC SET

1. Preliminaries. Let $\Phi(n, x)$ be the partial recursive function discussed by Kleene[2] which generates all partial recursive functions of one variable. We shall denote this function by $g_n(x)$. Following Rice[3] we use $g_n(x)$ to characterize r.e. sets. Let ω_n denote the range of $g_n(x)$, then $\{\omega_n\}$ is a sequence of r.e. sets in which every r.e. set occurs at least once. The set α is called *productive* if there exists a partial recursive function $p(n)$ such that $\omega_n \subset \alpha$ implies: (1) $p(n)$ is defined, (2) $p(n) \in \alpha - \omega_n$. Every such function $p(n)$ is called a *productive function* of α. Let Dom α denote the set of all n such that $\omega_n \subset \alpha$. The subset

[2] *Recursive predicates and quantifiers*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 41–73.

[3] *Classes of recursively enumerable sets and their decision problems*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 358–366.

π of α is called a *productive center* of the productive set α, if $\pi = p$ (Dom α) for some productive function $p(n)$ of α. The set α is called *productive in the sense of Post* (abbreviated: *P-productive*), if at least one of its productive functions is recursive. The set α is called *creative* (or *P-creative*), if α is r.e. and α' productive (respectively P-productive). Clearly, every P-productive set is productive and every P-creative set is creative.

The class of all creative sets is denoted by H. The question arises whether H and Z exhaust $F-E$. It is the purpose of this note to answer this question in the negative. The set α is called *medial* if it is not r.e. and neither immune nor productive. The set α is called *mesoic* if α is r.e. and α' medial, i.e., if $\alpha \in (F-E)-(Z+H)$.

We recall that α is many-one reducible to β [denoted by: α (m-1)red β], if there exists a recursive function which maps α into β and α' into β'; α is one-one reducible to β [denoted by: α (1-1)red β], if there exists a 1-1 recursive function which maps α into β and α' into β'. The degree of unsolvability of α relative to $(m-1)$ reducibility [or (1-1) reducibility] is denoted by $\Delta(\alpha)$ [respectively by: $d(\alpha)$]. If α (m-1)red β we write $\Delta(\alpha) \leqq \Delta(\beta)$; if α (m-1)red β is true, but β (m-1)red α is false, we write $\Delta(\alpha) < \Delta(\beta)$. Similarly $d(\alpha) \leqq d(\beta)$ and $d(\alpha) < d(\beta)$ are defined.

The following theorems[4] will be used.

A. If $f(x)$ is a partial recursive function defined for at least one value of x, we can effectively find a recursive function whose range is the same as that of $f(x)$.

B. The set α is productive if and only if $\alpha \neq o$ and there exists a partial recursive function $p(n)$ such that $\omega_n \neq o$ and $\omega_n \subset \alpha$ imply: (1) $p(n)$ is defined, (2) $p(n) \in \alpha - \omega_n$.

REMARK. Theorem B remains valid if we replace "productive" by "P-productive."

THEOREM 2.1. *If α (m-1)red β and α is productive, then β is productive.*

PROOF. There exists a recursive function which maps α into β and α' into β', say $f(n)$. Suppose $\omega_n \neq o$. By Theorem A we can now from the function $g_n(x)$ effectively find a partial recursive function ranging over ω_n, say $d(x)$. By comparing $f(0), \cdots, f(k)$ with $d(0), \cdots, d(k)$ for $k=0, 1, \cdots$, we can effectively find a recursive function ranging over $f^{-1}(\omega_n)$. Thus $f^{-1}(\omega_n)$ is r.e. If $\omega_n \subset \beta$, we know $f^{-1}(\omega_n) \subset \alpha$ and by

[4] These theorems are discussed in the author's paper *Productive sets*, not yet published.

the productivity of α we can effectively find an element $a \in \alpha - f^{-1}(\omega_n)$. Then $f(a) \in \beta - \omega_n$. We conclude that β is productive.

REMARK. It easily follows from this proof that the theorem remains valid if we replace "productive" by "P-productive" at both of its occurrences.

THEOREM 2.2. *If α $(m\text{-}1)$red β, α is creative and β is r.e., then β is creative.*

PROOF. α' $(m\text{-}1)$red β' since α $(m\text{-}1)$red β. But α' is productive, hence β' is productive. We conclude that β is creative.

REMARK. This theorem remains valid if we replace "creative" by "P-creative" at both of its occurrences.

Post proved [RES p. 295] that the complete set κ is P-creative by showing that κ is r.e. and κ' P-productive. The theorem mentioned in the last remark enables us to give a different proof of this fact. Clearly κ is r.e. Let $\alpha =_{df} \hat{n}[n \in \omega_n]$; from Post's proof [RES pp. 291, 292] that $\alpha \in F - E$ it follows immediately that α' is P-productive with the identity function as one of its productive functions. Thus α is P-creative. Observe that α (1-1)red κ, since every r.e. set is one-one reducible to the complete set [RES p. 297]. Then κ is P-creative because κ is r.e. and there exists a P-creative set (namely α) which is many-one reducible to κ.

2. The main result.

THEOREM 2.3. *There exists a set ν such that*:

(1) *ν is mesoic.*

(2) $d(\zeta) < d(\nu) < d(\kappa)$.

(3) $\Delta(\zeta) = \Delta(\nu) < \Delta(\kappa)$.

PROOF. (1) Let α be the set of all even non-negative integers, β the set of all odd non-negative integers, and $z(n)$ a 1-1 recursive function ranging over ζ. Suppose α_1 is the range of the function $2 \cdot z(n)$ and $\alpha_2 =_{df} \alpha - \alpha_1$. Then $\epsilon = \alpha_1 + \alpha_2 + \beta$, where α_1, α_2, β are mutually disjoint. We shall prove that α_1 is mesoic. The fact that ζ' is immune implies that α_2 is immune, since the mapping $n \rightarrow 2n$ maps ζ' recursively and 1-1 on α_2. Clearly, α_1 is r.e. To complete the proof it is now sufficient to prove that α_1' is medial. Observe that $\alpha_1' = \alpha_2 + \beta$. First of all, $\alpha_2 + \beta$ is not r.e., for if it were $\alpha_2 = (\alpha_2 + \beta)\beta'$ would be r.e., while we know that α_2 is immune. Secondly $\alpha_2 + \beta$ is not immune because it includes the infinite r.e. set β. Now suppose $\alpha_2 + \beta$ were productive. Then we could effectively find an element $c_0 \in (\alpha_2 + \beta) - \beta = \alpha_2$, an element $c_1 \in (\alpha_2 + \beta) - (\beta + \{c_0\}) = \alpha_2 - \{c_0\}$, etc. Then the immune set α_2 would include the infinite r.e. set $\{c_0, c_1, \cdots\}$, which is impossible. Thus α_1' is medial and α_1 mesoic. From now on we shall denote the set α_1 by ν.

(2) $d(\nu) \leqq d(\kappa)$, since every r.e. set is 1-1 reducible to κ. But $d(\kappa) \leqq d(\nu)$ is impossible because of Theorem 2.2 and the fact that ν is not creative. Thus $d(\nu) < d(\kappa)$. It follows from the definition of ν that the 1-1 recursive function $f(n) = 2n$ maps ζ on ν and ζ' into ν'. Hence $d(\zeta) \leqq d(\nu)$. But $d(\nu) \leqq d(\zeta)$ is impossible, since ν' does include an infinite r.e. set (namely β), while ζ' does not. Thus $d(\zeta) < d(\nu)$.

(3) $d(\zeta) < d(\nu)$ implies $\Delta(\zeta) \leqq \Delta(\nu)$. Recall the definition of ν. Suppose b is any element of ζ'. Let $f(n) =_{df} n/2$ if n is even (i.e., for $n \in \alpha$) and $f(n) =_{df} b$, if n is odd (i.e., for $n \in \beta$). Then $f(n)$ is a recursive function which maps α_1 on ζ, α_2 on ζ', and β on $\{b\}$; hence $f(n)$ maps ν on ζ and ν' into ζ'. It follows that $\Delta(\nu) \leqq \Delta(\zeta)$. Consequently $\Delta(\zeta) = \Delta(\nu)$. $d(\nu) < d(\kappa)$ implies $\Delta(\nu) \leqq \Delta(\kappa)$. But $\Delta(\kappa) \leqq \Delta(\nu)$ is impossible because of Theorem 2.2 and the fact that ν is not creative. Thus $\Delta(\nu) < \Delta(\kappa)$. This completes the proof.

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