A CONNECTED COUNTABLE HAUSDORFF SPACE

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In Math. Ann. vol. 94 (1925) pp. 262–295, Urysohn gave an example of a connected Hausdorff space with only countably many points. Here is another.

EXAMPLE 1. The points of the space are the rational points in the plane on or above the x-axis. If (a, b) is such a point and $\epsilon>0$, $(a, b)+\{(r, 0) \mid \text{ either } |r-(a+b/3^{1/2})|<\epsilon \text{ or } |r-(a-b/3^{1/2})|<\epsilon\}$ is a neighborhood.

To construct geometrically a neighborhood with center at (a, b), consider an equilateral triangle with base on the x-axis and apex at (a, b). (If $b=0$, regard (a, b) as the triangle.) Then (a, b) plus all rational points on the x-axis whose distances from a base vertex of the triangle are less than ϵ is an ϵ-neighborhood with center at (a, b).

This space satisfies the Hausdorff axioms and has the property that for each pair of neighborhoods, there is a point common to their closures. Hence, the space is connected.

Although this space has a countable basis, it is not regular and hence not metric. Its dimension depends on the definition of dimension used. In the Menger-Urysohn sense (the dimension is defined inductively in terms of boundaries of open sets) the space is one-dimensional, and in the Lebesgue sense (the dimension is defined in terms of orders of coverings) it is infinite-dimensional.

EXAMPLE 2. We may enlarge our description to get a connected countable Hausdorff space of any positive dimension (even infinite) in the Menger-Urysohn sense. We give one of dimension two.

The points of the space are the rational points of Euclidean 3-space whose second and third coordinates are non-negative. If $(a, b, 0)$ is a point, an ϵ-neighborhood with center at $(a, b, 0)$ is $(a, b, 0)+\{(r, 0, 0) \mid \text{ either } |r-(a+b/3^{1/2})|<\epsilon \text{ or } |r-(a-b/3^{1/2})|<\epsilon\}$. If (a, b, c), $c\neq 0$, is a point, an ϵ-neighborhood with center at (a, b, c) is the sum of (a, b, c) and all ϵ-neighborhoods with centers at points $(a, r, 0)$ where either $|r-(b+c/2^{1/2})|<\epsilon$ or $|r-(b-c/2^{1/2})|<\epsilon$.

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