THE INTERVAL TOPOLOGY OF A LATTICE[1]

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In this paper we obtain a necessary condition that a lattice be Hausdorff in its interval topology. This condition, stated in Proposition 2, can be applied to show that the interval topology of a Boolean algebra is Hausdorff if and only if every element is over an atom and that an l-group need not be Hausdorff in its interval topology. The former supplies a more or less complete answer to problem 76 of Birkhoff [1] and the latter solves 104. In addition a necessary and sufficient condition is obtained for a point to be isolated in the interval topology, thus answering, in part, problem 21.

Frink [2] has defined the interval topology of a lattice (or partly ordered set) by taking as a sub-basis for the closed sets all finite $[a, b]$ and infinite $[-\infty, a]$, $[a, \infty]$ closed intervals. A basis for the closed sets is then the collection of all finite unions of such intervals. As usual we say that a space is Hausdorff if for any two distinct points x and y, there exist disjoint open sets U and V with $x \in U$ and $y \in V$. Furthermore it is easily seen that we may select U and V from any given basis of open sets. Looking at the complements of U and V we obtain the dual requirement that given any two distinct points, the space can be covered by two closed sets each of which contains exactly one of the points, and in addition we may select these sets from any given basis for the closed sets. In particular:

PROPOSITION 1. *The interval topology of a partly ordered set is Hausdorff if and only if given any two distinct points there is a covering of the set by means of a finite number of closed intervals such that no interval contains both points.*

Now in a lattice the intersection of two closed intervals (finite or infinite) is empty or is a closed interval. For example, if $a \leqq x \leqq b$ and $c \leqq x \leqq d$, then $a \cup c \leqq x \leqq b \cap d$. In other words $[a, b] \cap [c, d] = [a \cup c, b \cap d]$. The same reasoning applies to intervals with infinite end points. To obtain a necessary condition that the interval topology of a lattice be Hausdorff, we look at any pair of comparable elements, $x < y$, for which by Proposition 1 there is a covering of the lattice by a finite number of closed intervals such that no interval contains $[x, y]$. Taking the trace on $[x, y]$ of each member of the covering we

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obtain a covering of $[x, y]$ by a finite number of closed subintervals, no one of which is $[x, y]$ itself. In other words, if we exclude x and y, each point of $[x, y]$ is comparable with at least one of the remaining end points of the subintervals. The same is true if either x or y is infinite. Let us say that a collection of elements a_i is a separating set of the interval $[x, y]$ if $x<a_i<y$ for each a_i and every element of $[x, y]$ is comparable with at least one of the a_i. If y covers x we will agree that the empty set separates $[x, y]$. Summarizing we have:

PROPOSITION 2. *A necessary condition for the interval topology of a lattice to be Hausdorff is that every closed interval have a finite separating set* (*fss*).

We are now in a position to prove:

PROPOSITION 3. *In a Boolean algebra without atoms, the interval* $[0, I]$ *has no fss.*

PROOF. If $\{a_1, a_2, \cdots, a_n\}$ is a fss, adjoin the complements of the a_i, obtaining a new set B. For each subset of B form the meet of its elements and from this collection of meets let $c_1, c_2, \cdots, c_k$ be the non-null minimal ones. It is convenient to think in terms of sets in which case the c_i are a collection of disjoint sets whose union intersects each a_i and its complement. Now for each c_i choose d_i so that $0<d_i<c_i$ and let $d=d_1\cup d_2\cup \cdots \cup d_k$. Then since $a_i>d\cap a_i$ we have $d\ngeq a_i$, and since $d\cap a_i'>0$, $d\nleq a_i$. In other words d is not comparable with any a_i.

REMARK. The lattice of all measurable subsets (modulo sets of measure zero) of the unit interval is a complete Boolean algebra without atoms, so its interval topology is, by the preceding theorem, not Hausdorff whereas the order topology is. Thus Proposition 3 may be applied to the solution of problem 76 of [1]. This problem has already been solved by B. C. Rennie [3] using a different method.

We might observe further that an examination of the proof of Proposition 3 shows that the following somewhat more general result may be established:

PROPOSITION 4. *A distributive lattice without atoms, in which each element* (*except I*) *has a non-null disjoint element, is not Hausdorff in its interval topology.*

PROPOSITION 5. *The interval topology of a Boolean algebra is Hausdorff if and only if every element is over an atom*[2].

[2] We have recently learned that this result has been obtained by M. Katetov, *Remarks on Boolean algebras*, Colloquium Math. vol. II 3–4 (1951).

PROOF. If some element x is over no atom, then the interval $[0, x]$ is a Boolean algebra without atoms. Hence, by Proposition 3, it has no fss and thus from Proposition 2 the topology is not Hausdorff. Assume then that every element is over an atom and let x and y be any pair of distinct elements. Since $x \cap y'$ and $y \cap x'$ cannot both be null there must be an atom a under, say, x but not y. It follows at once that the intervals $[a, I]$ and $[0, a']$ are disjoint closed intervals which cover the algebra, and the topology is Hausdorff (Proposition 1).

Next we apply Proposition 2 to Problem 104 of [1], which should read: "Is any l-group a topological group and a topological lattice in its interval topology?" Since the interval topology is T_1 and a topological group is regular, the interval topology must be Hausdorff if the l-group is to be a topological group. Now the additive group of all continuous real-valued functions defined on the closed unit interval is an l-group using the natural ordering [1, p. 216]. If f_0 denotes the function $f(x)=0$ and f_1 denotes the function $f(x)=1$, we show that the interval $[f_0, f_1]$ has no fss. If $\{a_1 \cdots a_n\}$ were such a set, choose for each a_i some point x_i where $a_i(x_i) \neq 1$. Define a continuous function $a(x)$ to be 1 at each of the x_i and elsewhere to take on values between 0 and 1 so that its integral over the interval is less than that of any a_i. Clearly $a(x)$ is not comparable with any of the a_i. We have shown:

PROPOSITION 6. *An l-group need not be a topological group in its interval topology.*

Finally we find a necessary and sufficient condition for a point x to be isolated in the interval topology of a lattice L. This is part of Problem 21 of [1]. First suppose that $0 < x < I$. If x is isolated, then $L - x$ is a closed set and in fact must be the union of a finite number of closed intervals $\{I_1 \cdots I_n\}$. Let P denote the set of elements of L under x and take the trace of each I_k on P, which is a closed interval by some earlier remarks. From the set of upper end points of the traces select the maximal ones. These form a nonempty finite set $\{x_1 \cdots x_n\}$, each x_i is covered by x, and any element under x is under some x_i. The same argument can be applied to the set of elements over x. Looking at the lower end point of each I_k let us replace it by $-\infty$ (x) if it is under (over) x. Then if an upper end point is under (over) x replace it by x (∞). Having done this we have a covering of L by a finite number of closed intervals for which none of the end points (except possibly x, 0, or I) is comparable with x. In other words x belongs to a fss of L in which no other member is comparable with x, and we have shown the necessity of the conditions in the following

PROPOSITION 7. *The following conditions are necessary and sufficient for an element x to be isolated in the interval topology of a lattice L.*

(a) *x covers a finite number of elements and every element under x is under an element covered by x.*

(b) *x is covered by a finite number of elements and every element over x is over an element which covers x.*

(c) *x belongs to a fss of L in which no other member is comparable with x.*

It is easy to see that the above conditions are sufficient. If the fss is $\{x, a_1 \cdots a_k\}$ and if x covers $\{b_1 \cdots b_m\}$ and if x is covered by $\{c_1 \cdots c_n\}$, than $L-x$ is the union of the following intervals: $[-\infty, a_i]$ $[a_i, \infty]$ $[-\infty, b_i]$ $[c_i, \infty]$ for all permissible values of i. If x is 0 or I then clearly (b) or (a) is necessary and sufficient for x to be isolated. In conclusion it should be stated that there are lattices having a fss for each interval yet containing nonisolated elements satisfying (a) and (b) of Proposition 7. Furthermore the interval topology is not Hausdorff.

BIBLIOGRAPHY

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