A CHARACTERIZATION THEOREM FOR MONOTONE MAPPINGS

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1. **Introduction.** Our main result states that a mapping (i.e. continuous function) f on the 2-sphere S onto itself is monotone if and only if f has a continuous extension g which maps the interior of S homeomorphically onto itself.

The fact that the above condition is sufficient for f to be monotone has been proved by Floyd in [1]. The necessity will follow from results obtained by Fort in [2] and Youngs in [3].

2. **Notation and terminology.** If X is a topological space, $H(X)$ is the group of all homeomorphisms of X onto X. We topologize $H(X)$ with the compact-open topology.

We let P be a plane, and let S be the unit 2-sphere in Euclidean 3-space. Both $H(P)$ and $H(S)$ are topological groups (under the compact open topology). It has been proved in [2] that $H(P)$ is locally arcwise connected.

By a *curve* in a topological space Y we mean a continuous function on $[0, 1]$ into Y. If Φ is a curve and $0 \leqq t \leqq 1$, we denote the value of Φ at t by Φ_t. A curve Φ *joins* x to y in a set K if $\Phi_0 = x$, $\Phi_1 = y$, and $\Phi_t \in K$ for all t in $[0, 1]$. A curve Φ in a metric space has *diameter* less than ϵ if and only if the distance from Φ_t to Φ_s is less than ϵ for all t, s in $[0, 1]$.

We define $R(S)$ to be the set of all rotations of S, and we let I be the identity mapping of S onto S. We let $p = (0, 0, 1)$ and then define $F(S)$ to be the set of all functions $f \in H(S)$ for which $f(p) = p$. Both $R(S)$ and $F(S)$ are obviously subgroups of $H(S)$.

We let d be the usual metric for S, and for mappings f, g of S onto S we define $\rho(f, g) = \sup_{x \in S} d(f(x), g(x))$. It is well known that the topology induced on $H(S)$ by ρ is the compact-open topology, and hence ρ is an admissible metric for $H(S)$.

3. **A local connectedness theorem.**

THEOREM 1. *$H(S)$ is locally arcwise connected.*

PROOF. It is obvious that $R(S)$ is locally arcwise connected. We now prove that $F(S)$ is also locally arcwise connected. This is accomplished by exhibiting a homeomorphism from $F(S)$ onto $H(P)$.

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We let π be the stereographic projection from p of $S-\{p\}$ onto P. If $f\in F(S)$, we define $T(f)=\pi f\pi^{-1}$. It is easy to verify that T is a homeomorphism of $F(S)$ onto $H(P)$. Since $H(P)$ is locally arcwise connected, it follows that $F(S)$ is also locally arcwise connected.

In order to prove that $H(S)$ is locally arcwise connected, it is sufficient to prove that corresponding to each $\epsilon>0$ there exists a $\delta>0$ such that if $f\in H(S)$ and $\rho(f, I)<\delta$ then f and I may be joined in $H(S)$ by a curve of diameter less than ϵ.

Suppose $\epsilon>0$. There exists $\alpha>0$ such that if ϕ and ψ are members of $H(S)$ which are in the α-neighborhood of I, then $\phi\psi$ is in the $\epsilon/2$-neighborhood of I. There exists $\beta>0$ such that: if $g\in R(S)$ and $\rho(g, I)<\beta$, then g and I can be joined in $R(S)$ by a curve of diameter less than α; and if $h\in F(S)$ and $\rho(h, I)<\beta$, then h and I can be joined in $F(S)$ by a curve of diameter less than α. We define $\delta=\beta/2$.

Suppose $f\in H(S)$ and $\rho(f, I)<\delta$. There exists $g\in R(S)$ such that $\rho(g, I)<\delta$ and $gf(p)=p$. Hence $gf\in F(S)$. It is easily seen that $\rho(gf, I)<2\delta=\beta$, and that $\rho(g^{-1}, I)<\beta$. Therefore there exists a curve Φ of diameter less than α which joins I to g^{-1} in $R(S)$, and there exists a curve Ψ of diameter less than α which joins I to gf in $F(S)$. We define a curve Ω by letting $\Omega_t=\Phi_t\Psi_t$ for $0\leqq t\leqq 1$. It is easily seen that Ω is a curve of diameter less than ϵ which joins I to f in $H(S)$. Thus $H(S)$ is locally arcwise connected at I and, since $H(S)$ is a topological group, this implies that $H(S)$ is locally arcwise connected.

Corollary 1. *$H(S)$ is uniformly locally arcwise connected with respect to the metric ρ.*

Proof. We make use of the fact that $\rho(f, g)=\rho(fh, gh)$ for all f, g, and h in $H(S)$.

Suppose $\epsilon>0$. There exists $\delta>0$ such that if $h\in H(S)$ and $\rho(h, I)<\delta$ then h can be joined to I in $H(S)$ by a curve of diameter less than ϵ.

Suppose f and g are in $H(S)$ and $\rho(f, g)<\delta$. Then $\rho(fg^{-1}, I)<\delta$. There exists a curve Φ of diameter less than ϵ which joins I to fg^{-1} in $H(S)$. We define a curve Ψ by letting $\Psi_t=\Phi_t g$ for $0\leqq t\leqq 1$. Then Ψ is a curve of diameter less than ϵ which joins g to f in $H(S)$. It follows from the arcwise connectedness theorem that there exists an arc of diameter less than ϵ which joins g to f in $H(S)$. Thus $H(S)$ is uniformly locally arcwise connected.

We now let $M(S)$ be the set of all monotone mappings of S onto S, and metrize $M(S)$ by ρ.

Corollary 2. *$M(S)$ is uniformly locally arcwise connected with respect to the metric ρ.*

PROOF. Youngs has proved in [3] that $H(S)$ is dense in $M(S)$. Our result therefore follows readily from Corollary 1.

4. A characterization of monotone mappings of S onto S.

We let Q be the interior of S.

THEOREM 2. *A mapping f of S onto S is monotone if and only if there exists a continuous extension g of f such that g maps $S \cup Q$ onto itself and $g|Q$ is a homeomorphism of Q onto Q.*

PROOF. It follows immediately from a theorem due to Floyd (see [1, p. 228]) that if such a g exists, then f is monotone.

Now suppose that f is monotone. We consider the space $M(S)$ metrized by ρ. Young's approximation theorem states that f is a limit point of $H(S)$. Since $H(S)$ is uniformly locally arcwise connected, it follows that there exists an arc Φ such that $\Phi_1 = f$ and $\Phi_t \in H(S)$ for $0 \leqq t < 1$. Each point of $S \cup Q$ can be represented in the form tx where $x \in S$ and $0 \leqq t \leqq 1$. We define $g(tx) = t\Phi_t(x)$ for all $x \in S$ and $0 \leqq t \leqq 1$. It is easily seen that g is a continuous function on $S \cup Q$ onto $S \cup Q$, that $g|S = f$, and that $g|Q$ is a homeomorphism of Q onto Q.

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