point so that instead of $U\cap A$ and $V\cap B$ one takes any subarc of A in U containing a and any subarc of B in V containing b. These changes are felt to be undesirable in being too great a departure from the original concept of $D(S)$ and in excluding sets with nonempty interiors.

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THE DEGREE FORMULA FOR THE SKEW-REPRESENTATIONS OF THE SYMMETRIC GROUP[1]

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1. **Introduction.** In his paper on the representations of the symmetric group,[2] G. de B. Robinson defines certain "skew-representations" and associates these to skew-diagrams (to be defined below) analogously to the way the irreducible representations of the symmetric group are associated with regular diagrams. Furthermore he shows that the degree of such a skew-representation is equal to the number of orderings of the related skew-diagram.[3]

The object of this note is to derive a formula for the degree of skew-representation related to a given skew-diagram.[4] This problem will be treated strictly in terms of the number of orderings of such a diagram, and from this point of view is very similar to the question attacked in [5] by R. M. Thrall.

In §4, this formula is applied to the problem of computing the characters of certain classes of the symmetric group.

2. **Definitions and lemmas.** A partially ordered set P is said to be *regular* or a *regular diagram* if:

(I) The elements of P may be represented by ordered pairs of integers (i, j), $i>0$, $j>0$, where $(i, j)\leqq(p, m)$ if and only if $i\leqq p$ and $j\leqq m$, $(i, j)=(p, m)$ if and only if $i=p$ and $j=m$,

(II) $\max_i (i, j)\leqq\max_i (i, j')$ whenever $j\geqq j'$,

(III) $(i, k)\in P$ implies $(j, k)\in P$ for all integers j with $1\leqq j\leqq i$,

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[2] See [1; 2; 3; 4].

[3] See [1, p. 290].

[4] This is an answer to the question raised in [1, p. 294], the ϕ of that paper is the g of the theorem below.

(IV) If P is nonempty, $(1, 1)\in P$ and $(1, 1)\leqq p$ for all $p\in P$.

If P is regular and has k rows with a_j elements in the jth row, P will be denoted by $[a_1, \cdots, a_k]$; clearly $a_1\geqq a_2\geqq \cdots \geqq a_k>0$. If $Q=[b_1, \cdots, b_k]$, with $b_i\leqq a_i$, $i=1, \cdots, k$, then $Q\subset P$ and $P-Q=[a_1, \cdots, a_k]-[b_1, \cdots, b_k]$ is the ordinary set theoretic difference, a partially ordered set of this form will be called *skew* or a *skew diagram.*

A one to one order preserving mapping of a regular or skew-diagram onto a linearly ordered set will be called an *ordering*. The number of orderings of a skew-diagram $[a_1, \cdots, a_k]-[b_1, \cdots, b_k]$ will be denoted by $g([a_1, \cdots, a_k]-[b_1, \cdots, b_k])$ if $b_r=b_{r+1}=\cdots=b_k=0$. This will simply be written as $g([a_1, \cdots, a_k]-[b_1, \cdots, b_{r-1}])$; if $r=1$, this will be written as $g([a_1, \cdots, a_k])$.

Before proceeding with a proof of the formula for $g(K)$ where $K=[a_1, \cdots, a_k]-[b_1, \cdots, b_k]$, $a_i\geqq b_i$, $i=1, \cdots, k$, it is necessary to note the following relations, conventions, and lemmas.

The only possible first elements in any ordering of $K=[a_1, \cdots, a_k]-[b_1, \cdots, b_k]$ are elements of the form (b_i+1, i); an element of this type is first in some ordering if and only if $(b_i+1, i)\in K$ and $(b_{i-1}+1, i-1)\notin K$, which is equivalent to saying that $b_i\neq b_{i-1}$, $b_i\neq a_i$, hence

$$(1)\qquad \begin{aligned} &g([a_1, \cdots, a_k] - [b_1, \cdots, b_k]) \\ &\qquad = \sum g([a_1, \cdots, a_k] - [b_1, \cdots, b_j + 1, \cdots, b_k]) \end{aligned}$$

where the summation is taken over all j with $b_j\neq b_{j-1}$, and $b_j\neq a_j$. This simply states that the number of orderings of K is the sum of the number of orderings of $K-\{x\}$ as x ranges over all possible first elements.

The convention will be made throughout this paper that $1/r!=0$ if $r<0$.

If A is a determinant, then the minor of A with the pth row and mth column deleted will be denoted by A_{pm}. The following lemma is obvious.

LEMMA 1. *If* $A=\det(a_{ij})$, $i, j=1, \cdots, k$, *and* $A^{(t)}=\det(a_{ij}^{(t)})$ *where* $a_{ij}^{(t)}=a_{ij}$ *if* $i\neq t$ *and* $a_{tj}^{(t)}=r_j a_{tj}$, $j=1, \cdots, k$, *then* $\sum_{t=1}^{k} A^{(t)}=A(\sum_{j=1}^{k} r_j)$.

LEMMA 2. *If there exists a* t *such that* $a_{ij}=0$ *for* $i\leqq t\leqq j$, *then* $A=\det(a_{ij})=0$.

The proof is by induction on k, the dimension of (a_{ij}). The lemma is clearly true for $k=1$. Now assume it true for $k-1$, then

$$A = \sum_{j=1}^{k} (-1)^{j+1} a_{1j} A_{1j} = \sum_{j=1}^{t-1} (-1)^{j+1} a_{1j} A_{1j},$$

as $a_{1j}=0$ for $j \geqq t$. Each determinant A_{1j}, $j<t$, is of the above form and hence, by the induction hypothesis, $A_{1j}=0$, $j<t$, therefore $A=0$.

3. The formula for skew-diagrams.

THEOREM. $g([a_1, \cdots, a_k]-[b_1, \cdots, b_k])=n! \det (z_{ij})$, *where* $n=\sum_{i=1}^{k} (a_i-b_i)$, $z_{ij}=1/(a_j-b_i-j+i)!$.

The proof is by induction on n. Let $n=1$. It is sufficient to show that $\det (z_{ij})=1$. There exists a t such that $a_t=b_t+1$ and $a_i=b_i$, for $i \neq t$, therefore $z_{tt}=1/(a_t-b_t)!=1/1!=1$, and $z_{ii}=1/(a_i-b_i)!=1/0!=1$ for $i\neq t$.

For $j>i\neq t$, $(a_j-b_i-j+i)=(a_j-a_i-j+i) \leqq (-j+i)<0$, therefore $z_{ij}=1/(a_j-b_i-j+i)!=0$. For $j>i=t$, $(a_j-b_t-j+t)=(a_j-a_t-j+t+1) \leqq (-j+t+1) \leqq 0$. $(a_j-b_t-j+t)=0$ if and only if $a_j=a_t$ and $j=t+1$, therefore $a_t=a_{t+1}$, but then $b_t+1=a_t=a_{t+1}=b_{t+1}$, and hence $b_t<b_{t+1}$, which is impossible; therefore $(a_j-b_t-j+t)<0$, which implies $z_{tj}=1/(a_j-b_t-j+t)!=0$ for $j>t$. Combining these results yields $z_{ii}=1$, $z_{ij}=0$ for $j>i$, which shows that

$$\det (z_{ij}) = \prod_{i=1}^{k} z_{ii} = 1.$$

Now assume the theorem true for $n-1$. $g([a_1, \cdots, a_k]-[b_1, \cdots, b_t+1, \cdots, b_k])=(n-1)! \det (z_{ij}^{(t)})$ by the induction hypothesis where $z_{ij}^{(t)}=z_{ij}$ if $i\neq t$ and $z_{tj}^{(t)}=1/(a_j-b_t-j+t-1)!$.

If $b_t=b_{t-1}$, $z_{t-1,j}^{(t)}=1/(a_j-b_{t-1}-j+t-1)!=1/(a_j-b_t-j+t-1)!=z_{tj}^{(t)}$, therefore $\det (z_{ij}^{(t)})=0$ as two columns are identical. Let $a_t=b_t$; if $i<t\leqq j$, then $z_{ij}^{(t)}=z_{ij}=1/(a_j-b_i-j+i)!=0$ because $(a_j-b_i-j+i) \leqq (a_t-b_t-j+i)=(-j+i)<0$. If $i=t\leqq j$, then $z_{tj}^{(t)}=1/(a_j-b_t-j+t-1)!=0$ because

$$\begin{aligned}(a_j - b_t - j + t - 1) &= (a_t - b_t - j + t - 1) \\ &= (-j + t - 1) \leqq -1 < 0.\end{aligned}$$

Therefore for $i\leqq t\leqq j$, $z_{ij}^{(t)}=0$ and hence, by Lemma 2, $\det (z_{ij}^{(t)})=0$. From the cases just examined above it follows that

$$\begin{aligned}&g([a_1, \cdots, a_k] - [b_1, \cdots, b_k]) \\ (2)\qquad &= \sum_{t=1}^{k} g([a_1, \cdots, a_k] - [b_1, \cdots, b_t + 1, \cdots, b_k])\end{aligned}$$

as only zero has been added to the right-hand side of (1).

By the induction hypothesis and (2),

$$g([a_1, \cdots, a_k] - [b_1, \cdots, b_k]) = (n-1)! \sum_{t=1}^{k} \det (z_{ij}^{(t)})$$

where $z_{ij}^{(t)} = z_{ij}$ for $i \neq t$ and

$$z_{tj}^{(t)} = 1/(a_j - b_t - j + t - 1)! = (a_j - b_t - j + t)/(a_j - b_t - j + t)!$$
$$= (a_j - j)z_{tj} + (t - b_t)z_{tj}.$$

Therefore by Lemma 1

$$\sum_{t=1}^{k} \det (z_{ij}^{(t)}) = \det (z_{ij}) \left(\sum_{t=1}^{k} t - \sum_{t=1}^{k} b_t + \sum_{i=1}^{k} a_i - \sum_{j=1}^{k} j \right)$$
$$= n \det (z_{ij}),$$

from which it follows that

$$g([a_1, \cdots, a_k] - [b_1, \cdots, b_k]) = n! \det (z_{ij})$$

and the thorem is proved.

4. **An application.**[5] As an application to the theory of characters of the symmetric group, the following formula is proved:

THEOREM. *Let χ^λ denote the irreducible character associated with the regular diagram λ; if σ is a permutation in $\mathfrak{S}_{m+n}$ which leaves the last n letters fixed, then*

(3) $$\chi^\lambda(\sigma) = \sum \chi^\mu(\sigma) g(\lambda - \mu)$$

where the summation ranges over all regular diagrams μ containing m points.

PROOF. Let $\chi^{\lambda-\mu}$ denote the (reducible) representation associated with the skew-diagram $\lambda - \mu$.[6] Robinson has shown[7] that if a permutation in $\mathfrak{S}_{m+n}$ is of the form $\sigma_1\sigma_2$, where σ_1 acts on the first m letters, σ_2 on the last n letters, then

(4) $$\chi^\lambda(\sigma_1\sigma_2) = \sum \chi^\mu(\sigma_1)\chi^{\lambda-\mu}(\sigma_2)$$

where the summation is over all regular diagrams μ on m letters contained in the regular diagram λ.

[5] This was suggested by Professor Robinson and Thrall and relates this result to a forthcoming paper by J. S. Frame, R. M. Thrall, and G. de B. Robinson.

[6] See [1], especially pp. 289–290.

[7] See [1].

The function $g(\lambda-\mu)$ evaluated in §3 is the degree of the character $\chi^{\lambda-\mu}$. Hence if σ_2 is the identity permutation, formula (4) reduces to

$$\chi^{\lambda}(\sigma_1) = \sum \chi^{\mu}(\sigma_1)g(\lambda - \mu). \tag{5}$$

In the proof of the main theorem in §3 it was shown that $g(\lambda-\mu)=0$ if μ is not contained in λ, therefore the sum may be extended over all regular diagrams μ containing m points.

If for example σ is a transposition then $\chi^{[2]}=1$, $\chi^{[1^2]}=-1$ and hence $\chi^{\lambda}(\sigma)=g(\lambda-[2])-g(\lambda-[1^2])$, for any regular diagram λ.

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