ON THE ALGEBRAIC STRUCTURE OF DISCONTINUOUS GROUPS

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The purpose of this paper is to determine what may be said, under favorable conditions, about the algebraic structure of a group of homeomorphisms when a fundamental domain is known.

If S is a space and G a group of homeomorphisms acting on S, then the image of a point x in S under an element g in G will be denoted by gx. If D is a subset of S, then gD will be the image of D, and if F is a subset of G, then Fx will denote the union of the images gx of x and FD will denote the union of the sets gD for all g in F. The empty set will be denoted by $\emptyset$, the identity element of a group by 1, and intersections, inclusions, and closures will be indicated in the usual ways.

Let there be given a connected and locally connected Hausdorff space S, a group G of homeomorphisms acting on S, and a connected subspace D of S satisfying the following conditions:

1. S is the disjoint union of the sets gD with $g \in G$.
2. Let $F = \{g \in G \mid g\bar{D} \cap \bar{D} \neq \emptyset\}$. Then the number of elements in F is finite and FD contains a neighborhood of D.

If the foregoing conditions are satisfied, then in order that these concepts have names, S will be called an admissible space, G a group of the first kind acting on S, and D a proper fundamental domain for G. It can be verified without great difficulty that the elements of F generate G,[2] and F will be called the local set of generators relative to D. D will be fixed throughout, and the phrase "relative to D" will generally be omitted. It is evident that $1 \in F$ and that $g \in F$ implies $g^{-1} \in F$. If the space S is simply connected, then the relations which hold between the local generators can be found. To state the main theorem precisely it is necessary to introduce certain groups of which G is a homomorphic image.

Let $\tilde{H}$ be an abstract free group the generators of which are the elements of F different from the identity. To avoid confusion it will be assumed that a mapping σ is given which takes the elements of F different from the identity, considered as a subset of G, onto the generators of $\tilde{H}$, and that σ is extended to all of F by setting $\sigma(1) = 1$. In $\tilde{H}$ form the smallest normal subgroup R containing all the elements

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[2] The demonstration is omitted because it is already present in the literature; cf. C. L. Siegel [1].

$\sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1}$ with g_1, g_2, and g_1g_2 all in F, and let H denote the quotient group of $\tilde{H}$ modulo R. The mapping σ of F into $\tilde{H}$ followed by the homomorphism of $\tilde{H}$ onto H produces a mapping of F into H which is again one-to-one. Since $\tilde{H}$ will be little used in the remainder of this paper, the latter mapping will also be denoted by σ. Then in H it is the case that $\sigma(g_1)\sigma(g_2)=\sigma(g_1g_2)$ whenever g_1, g_2, and g_1g_2 are in F. Letting $g_2=g_1^{-1}$, it follows that $\sigma(g^{-1})=\sigma(g)^{-1}$.

There is a natural homomorphism of $\tilde{H}$ onto G defined by taking the elements of F considered as generators of $\tilde{H}$ back to the same elements considered as elements of G. The elements of the kernel of this homomorphism will be called the relations of G. The kernel contains R and the homomorphism in question induces a natural homomorphism of H onto G, which will be denoted by ϕ. If σ is considered as a mapping of F into H (it will always be so considered henceforth), then σ followed by ϕ is the identity mapping on F. The elements of the group R will be called the local relations. H itself will be called the local universal covering group of G and ϕ will be called the local covering homomorphism. This terminology may be partly justified by the main theorem which can now be stated.

THEOREM. *Let there be given an admissible space S, group of the first kind G acting on S, and proper fundamental domain D for G, and let H be the local universal covering group of G and ϕ the local covering homomorphism. Then there exists an admissible (hence in particular, connected) space T with the properties:*

1. *T is a covering space of S. The covering mapping will be denoted by ψ.*

2. *H (or more properly, a group isomorphic to H) acts as a group of homeomorphisms of the first kind on T and has a proper fundamental domain E homeomorphic to D; in fact ψ is a homeomorphism on a neighborhood of E and $\psi(E)=D$.*

3. *If y is a point of T and h an element of H, then $\psi(hy)=\phi(h)\psi(y)$. There is commutativity in the diagram:*

$$\begin{array}{ccc} T & \xrightarrow{\;h\;} & T \\ \psi\downarrow & & \downarrow\psi \\ S & \xrightarrow{\;\phi(h)\;} & S \end{array}$$

The following corollary is an immediate consequence.

COROLLARY. *If S is simply connected, then G is isomorphic to H; all the relations of G are local.*

PROOF OF THE MAIN THEOREM. (The proof will contain five lemmas.) The set F of local generators being as before, let FD (which may be called the star of D) be denoted by D^*. For every element h of H take a copy of the fundamental domain D and let this copy be denoted by hE. The copy corresponding to the identity of H may be denoted simply by E. Let the union of the sets hE be denoted by T. T, when suitably topologized, will be the space desired.

If x is a point of D, let hx denote the corresponding point in the copy hE of D. H acts as a group of one-to-one transformations on T by left multiplication. The homomorphism ϕ induces a mapping ψ of T onto S defined by $\psi(hx)=\phi(h)x$. Conclusion 3 of the main theorem follows automatically. T will be topologized so that ψ is a covering mapping.

The set $\sigma(F)E$ will be denoted by E^*. Since σ followed by ϕ is the identity on F, it follows that ψ is one-to-one on any set of the form hE^*. Define a subset of hE^* to be open if its image under ψ is open and let these open sets form a subbase for the open sets of T; ψ is then continuous and H a group of the first kind with proper fundamental domain E. Since D^* was assumed to contain a neighborhood of D, this does define a topology in T. From the definition of ψ, however, it is evident only that ψ is continuous but not that it is open. The next few lemmas are preparatory to the proof of that fact.

LEMMA 1. *If h_1 and h_2 are in H, then $\psi(h_1E^*\cap h_2E^*)$ is relatively open in $\psi(h_1E^*)\cap\psi(h_2E^*)$.*

PROOF. Let $\psi(h_1E^*)\cap\psi(h_2E^*)$ be denoted by X and $\psi(h_1E^*\cap h_2E^*)$ be denoted by Y. It will be proved by contradiction that $X-Y$ is relatively closed in X.

Suppose x is in the closure in X of $X-Y$ but not in $X-Y$. Now, X, Y, and $X-Y$ are contained in $\phi(h_1)D^*$ and are each disjoint unions of a finite number of images gD of D under elements g of F. Since x is in the closure of $X-Y$, x must already be in the closure of some gD contained in $X-Y$ because of the finiteness of the number of these. Since this gD is contained in X it must be a set of the form $\phi(h_1)g_1D$ which is also of the form $\phi(h_2)g_2D$, where g_1 and g_2 are in F. Therefore (1): $\phi(h_1)g_1=\phi(h_2)g_2$ since D is a fundamental domain for G. It must be that (2): $h_1\sigma(g_1)\neq h_2\sigma(g_2)$ or gD would be in Y contrary to assumption. Note again that x is in $\phi(h_1)g_1\overline{D}$.

But x was assumed to be not in $X-Y$, which means that x is in Y. Being in X, it is therefore in a set of the form $\phi(h_1)g_3D$ which is also of the form $\phi(h_2)g_4D$ implying as before that (3): $\phi(h_1)g_3=\phi(h_2)g_4$, where in addition, (4): $h_1\sigma(g_3)=h_2\sigma(g_4)$ in order that x may be in Y.

Since x was in both $\phi(h_1)g_1\bar{D}$ and $\phi(h_1)g_3D$, it follows that $\phi(h_1)g_1\bar{D} \cap\phi(h_1)g_3D\neq\emptyset$, whence $g_1\bar{D}\cap g_3D\neq\emptyset$. Therefore $g_1^{-1}g_3$ is in F. But from (1) and (3) we have $g_1^{-1}g_3=g_2^{-1}g_4$. Therefore (5): $\sigma(g_1)^{-1}\sigma(g_3) =\sigma(g_1^{-1}g_3)=\sigma(g_2^{-1}g_4)=\sigma(g_2)^{-1}\sigma(g_4)$, by the manner in which H was constructed. Combining (5) with (2) one obtains that $h_1\sigma(g_3) \neq h_2\sigma(g_4)$, which contradicts (4). This proves the lemma.

LEMMA 2. *Let there be given sets T, S, a mapping $\psi: T\to S$, and subsets N_1, O_1, N_2, O_2 of T such that $N_1\supset O_1$ and $N_2\supset O_2$. Suppose that the restriction of ψ to N_1 is one-to-one and that the restriction of ψ to N_2 is likewise one-to-one. Then $\psi(O_1\cap O_2)=\psi(O_1)\cap\psi(O_2)\cap\psi(N_1\cap N_2)$.*

The proof of this lemma is not difficult and is omitted. Lemma 2 was phrased for arbitrary sets and a mapping between them; now let T, S, and ψ be as before.

LEMMA 3. *$\psi: T\to S$ is an open mapping.*

PROOF. On sets of the form hE^*, ψ is one-to-one, and a subbase for the open sets of T was obtained by letting a subset of hE^* be open if its image under ψ was open. Therefore ψ maps an open set of the given subbase onto an open set. It must be shown that ψ does the same to intersections of subbasic open sets. For this it is sufficient to prove the following: If O_1 is an open set of T contained in a set of the form h_1E^* and O_2 an open set of T contained in a set of the form h_2E^*, and if $\psi(O_1)$ and $\psi(O_2)$ are both open, then $\psi(O_1\cap O_2)$ is also open. Since ψ is one-to-one on sets of the form hE^*, Lemma 2 may be applied and one obtains $\psi(O_1\cap O_2)=\psi(O_1)\cap\psi(O_2)\cap\psi(h_1E^*\cap h_2E^*)$. Since $\psi(O_1)$ and $\psi(O_2)$ are open, it follows from Lemma 1 that this is relatively open in $\psi(O_1)\cap\psi(O_2)\cap\psi(h_1E^*)\cap\psi(h_2E^*)$. But this is just $\psi(O_1) \cap\psi(O_2)$, since $O_1\subset h_1E^*$ and $O_2\subset h_2E^*$. Therefore $\psi(O_1\cap O_2)$ is relatively open in $\psi(O_1)\cap\psi(O_2)$. But the latter is an open set, so it follows that the former is also, which proves the lemma.

From the fact that ψ is continuous, open, and one-to-one on sets of the form hE^* it follows that ψ is a homeomorphism on such sets; conclusion 2 of the main theorem follows from this.

Nothing so far proved implies that T is a Hausdorff space. This could be shown directly, but will also follow automatically from the fact that T covers S.

LEMMA 4. *$\psi: T\to S$ is a covering mapping.*

PROOF. Let a point x in S be given; we may assume it is in D. It must be shown that there exists a neighborhood U of x such that the inverse image under ψ of U is a disjoint union of open sets each of which is mapped homeomorphically onto U by ψ. But there exists a neighborhood U of x which is contained in D^*, since it was assumed that D^* contains in fact a neighborhood of D. It will be shown that such a U will serve. Let K denote the kernel of ϕ; K will be the group of covering homeomorphisms.

For every h_i in K, form the set $\psi^{-1}(U)\cap h_iE^*$, which will be denoted by U_i. U_i is open since it is contained in a set of the form hE^* and its image under ψ, which is U, is open; all such sets were in the subbasis for the open sets of T by means of which the topology of T was defined. Furthermore ψ maps U_i homeomorphically on U. The inverse image of U under ψ is the union of the sets U_i and it is therefore sufficient to prove that the sets U_i are disjoint. Suppose not: say $U_1\cap U_2 \neq \emptyset$. U_1 and U_2 each contain exactly one inverse image of x. Let these points be called y_1 and y_2 respectively, and let z be a point in $U_1\cap U_2$. Then z is in a set of the form $h_1\sigma(g_1)E$ which is also of the form $h_2\sigma(g_2)E$, where g_1 and g_2 are in F. This implies that h_1E and h_2E are both in $h_1\sigma(g_1)E^*=h_2\sigma(g_2)E^*$. But ψ can not then be a homeomorphism on $h_1\sigma(g_1)E^*$, for this set contains both y_1 and y_2, which have the same image. This contradiction shows that the sets U_i are indeed disjoint and ends the proof of the lemma. The assumption that S is locally connected has tacitly entered here; it is otherwise not even meaningful to speak of a covering space of S.

Lemma 4 proves conclusion 1 of the main theorem. It remains only to prove that T is an admissible space. Since ψ is a local homeomorphism, it is evident that T will be localled connected. All that remains, then, is to show that T is connected.

LEMMA 5. *T is connected.*

PROOF. D is connected and ψ is a homeomorphism on sets of the form hE^* with h in H. Therefore E and every set of the form hE with h in H is connected. If g is any element of F, then gD is by assumption adherent to D. Therefore $\sigma(g)hE$ is adherent to hE for any element h of H, again because ψ is a homeomorphism on sets of the form hE^*. Therefore, the connected component of a point of E must contain all of E, and with any set hE it must contain every set of the form $\sigma(g)hE$ with g in F. But $\sigma(F)$ generates H. Therefore the connected component of any point of E contains all of T, and so T must be connected. This proves the lemma.

With the demonstration of Lemma 5, the proof of the main theorem is complete.

While the space T whose existence is asserted in the main theorem is a covering space of S, it is in general not the universal covering but does have certain analogous properties. Let the triple consisting of an admissible space S, group G of the first kind acting on S, and a proper fundamental domain D, be called a localized discontinuous group of the first kind. For simplicity, the phrase "of the first kind" will be dropped, but "localized" is included to indicate that a specific fundamental domain is given. If $\mathfrak{G}=(S, G, D)$ and $\mathfrak{H}=(T, H, E)$ are two localized discontinuous groups, then $\mathfrak{H}$ will be said to be a covering group of $\mathfrak{G}$ if

1. T is a covering space of S, G is a homomorphic image of H, and there is a covering mapping ψ and a homomorphism ϕ satisfying 2 and 3.

2. There is a neighborhood of E on which ψ is a homeomorphism and $\psi(E)=D$.

3. If y is a point of T and h an element of H, then $\psi(hy)=\phi(h)\psi(y)$.

The pair (ψ, ϕ) will be called a covering mapping of $\mathfrak{H}$ onto $\mathfrak{G}$. Then the main theorem really asserts that if a localized discontinuous group $\mathfrak{G}=(S, G, D)$ is given, then there exists a covering group $\mathfrak{H}=(T, H, E)$ in which H is the local universal covering group of G. For $\mathfrak{G}$ and $\mathfrak{H}$ one may then assert the following.

THEOREM. *If $\mathfrak{G}'$ is any covering group of $\mathfrak{G}$, then $\mathfrak{H}$ is a covering group of $\mathfrak{G}'$.*

The proof of this theorem is not difficult and will be omitted. It indicates the sense in which H may be said to be universal and justifies calling $\mathfrak{H}$ the local universal covering group of G.

It is easy to give examples of the application of the main theorem. In theory, at least, it gives the structure of most of the common discontinuous groups of complex-analytic homeomorphisms operating on the upper half-plane, and in fact readily gives the structure of those groups whose fundamental domains are of a simple nature, such as the modular group.

REFERENCE

1. C. L. Siegel, *Discontinuous groups*, Ann. of Math. vol. 44 (1943).

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