ZEROS OF SELF-INVERSIVE POLYNOMIALS

GERMÁN ANCOCHEA

1. **Introduction.** In a recent paper [2] F. F. Bonsall and Morris Marden have given a new proof of Cohn's theorem [3] concerning the polynomials whose zeros are symmetric in the unit circle $C: |z| = 1$. The proof, though simpler than Cohn's, became complicated by the consideration of many cases according the behaviour of the polynomial on $|z| = 1$. In the present note we give another proof of Cohn's theorem sensibly simpler than the previous ones.

2. **Self-inversive polynomials.** Let $f(z)$ be a polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n.$$

We denote by $[f(z)]^*$ the polynomial

$$[f(z)]^* = z^n \cdot \overline{f\left(\frac{1}{\bar{z}}\right)} = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n,$$

inverse of $f(z)$, whose zeros are symmetric to $f(z)$'s zeros with respect to C. For every $f(z)$ one has, on $|z| = 1$,

$$|f(z)| = |[f(z)]^*|.$$

A polynomial

$$g(z) = b_0 + b_1 z + \cdots + b_m z^m \tag{2.1}$$

is said to be a self-inversive polynomial when

$$g(z) = c[g(z)]^*, \qquad |c| = 1. \tag{2.2}$$

Let $g(z)$ be the self-inversive polynomial (2.1) and let $g'(z)$ be its derivative. From (2.2) follows the identity

$$zg'(z) + c[g'(z)]^* = mg(z). \tag{2.3}$$

3. **Cohn's theorem.** *Let $g(z)$ be the self-inversive polynomial* (2.1). *Then $g(z)$ has the same number of zeros inside the unit circle C as does the polynomial*

$$c[g'(z)]^* = mb_0 + (m-1)b_1 z + \cdots + b_{m-1} z^{m-1}.$$

For the proof of the theorem we shall need only Rouché's theorem and the following lemma, whose indirect proof based on zero's continuity is immediate.

Received by the editors January 5, 1953.

LEMMA. *Let ϵ be a real positive number. If, for every λ such that $0<\lambda<\epsilon$, the polynomial*

$$f(z) + \lambda R(\lambda, z)$$

has a fixed number h of zeros inside C, the number of zeros of $f(z)$ inside C is $\leqq h$.

In [2], besides Rouché's theorem and the preceding lemma, essential use is made of another lemma which is a consequence of the following one given in [1]: *for every self-inversive polynomial $g(z)$ (2.1) one has, on $|z|=1$, $|g'(z)/g(z)| \geqq m/2$.*

4. **Proof of the theorem.** For the sake of brevity we shall denote by p and p_1 respectively the number of zeros inside C of $g(z)$ and $c[g'(z)]^*$.

(a) $p_1 \leqq p$. Let ϵ be a positive real number such that for $0<\lambda<\epsilon$ the polynomial

$$g((1-\lambda)z)$$

has inside C the same number p of zeros as does $g(z)$. By (2.2) we have

$$c[g((1-\lambda)z)]^* = (1-\lambda)^m g(z/(1-\lambda)).$$

On the other hand, Rouché's theorem assures that the polynomial

$$\begin{aligned} H(\lambda, z) &= g((1-\lambda)z) - (1-\lambda)^m c[g((1-\lambda)z)]^* \\ &= g((1-\lambda)z) - (1-\lambda)^{2m} g(z/(1-\lambda)) \end{aligned}$$

has also p zeros in $|z|<1$. We write $H(\lambda, z)$ in the form

$$H(\lambda, z) = H(0, z) + \lambda H_\lambda'(0, z) + \lambda^2 R(\lambda, z).$$

Since $H(0, z)=0$, one has

$$\begin{aligned} \frac{1}{\lambda} H(\lambda, z) &= H_\lambda'(0, z) + \lambda R(\lambda, z) = (-2zg'(z) + 2mg(z)) + \lambda R(\lambda, z) \\ &= 2c[g'(z)]^* + \lambda R(\lambda, z). \end{aligned}$$

Hence, from the lemma, $p_1 \leqq p$.

(b) $p \leqq p_1$. On $|z|=1$ we have

$$|zg'(z)| = |g'(z)| = |[g'(z)]^*| = |c[g'(z)]^*|.$$

Let $0<\lambda<1$. From Rouché's theorem it follows that the polynomial

$$c[g'(z)]^* + (1-\lambda)zg'(z) \tag{4.1}$$

has p_1 zeros in $|z|<1$. But, by (2.3), (4.1) can be written in the form

$$(c[g'(z)]^* + zg'(z)) - \lambda zg'(z) = mg(z) - \lambda zg'(z).$$

Then the lemma gives $p \leqq p_1$. This completes the proof of the theorem.

References

1. G. Ancochea, *Sur les polynomes dont les zéros son symétriques par rapport à une circonférence*, C. R. Acad. Sci. Paris vol. 221 (1945) pp. 13–15.

2. F. F. Bonsall and M. Marden, *Zeros of self-inversive polynomials*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 471–475.

3. A. Cohn, *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einer Kreise*, Math. Zeit. vol. 14 (1922) pp. 110–148.

University of Madrid

MEASURE EXTENSIONS AND THE MARTINGALE CONVERGENCE THEOREM[1]

SHU-TEH CHEN MOY

1. **Introduction.** In 1940 J. L. Doob proved the following martingale convergence theorem [3].[2]

Let $\{x_n, \mathcal{J}_n, n \geqq 1\}$ be a martingale[3] *with*

$$\sup \{E[|x_n|]: n \geqq 1\} < \infty.$$

Then $\{x_n\}$ converges with probability 1 *to a random variable x_∞ of finite expectation.*

In 1946 E. S. Andersen and B. Jessen proved some limit theorems on derivatives of set functions [1]. One of the theorems is closely related to the martingale convergence theorem and is stated below.

Let $\mathcal{J}_1 \subset \mathcal{J}_2 \subset \cdots \subset \mathcal{J}_n \subset \cdots$ be a nondecreasing sequence of Borel fields of subsets of a nonempty set Ω. Let P be a probability measure defined on the smallest Borel field $\mathcal{J}_\infty$ containing all the $\mathcal{J}_n$'s. Let ϕ be a bounded, countably additive set function defined on $\mathcal{J}_\infty$. Let P_n, ϕ_n be the contractions of P, ϕ to $\mathcal{J}_n$ respectively and suppose that each ϕ_n is absolutely continuous with respect to P_n. Let x_n be the derivative of ϕ_n relative to P_n. Then $\{x_n\}$ converges, except on a set of P measure 0, *to*

Presented to the Society, April 25, 1953; received by the editors April 15, 1953.

[1] This work was done while the author was Emmy Noether Fellow of Bryn Mawr College. It is based on a portion of the doctoral thesis submitted to the University of Michigan. The thesis was written under the supervision of Professor J. L. Doob of the University of Illinois.

[2] Numbers in brackets refer to the bibliography.

[3] For the definition and properties of a martingale see [2, Chap 7].