A GEOMETRICAL CONFIGURATION WHICH IS A PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN[1]

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1. Introduction. It will be shown that a geometrical configuration consisting of certain suitably chosen lines of the finite projective geometry $PG(3, s)$ may be interpreted as the partially balanced incomplete block design having parameters

$$v = b = (s^4 - 1)/(s - 1), \quad \lambda_1 = 1, \quad n_1 = s(s + 1),$$
$$r = k = s + 1, \quad \lambda_2 = 0, \quad n_2 = s^3,$$

$$(1.1) \qquad P_1 = (p^1_{jk}) = \begin{pmatrix} s - 1 & s^2 \\ s^2 & s^2(s - 1) \end{pmatrix},$$

$$P_2 = (p^2_{jk}) = \begin{pmatrix} s + 1 & s^2 - 1 \\ s^2 - 1 & s^2(s - 1) \end{pmatrix},$$

where $s = p^n > 1$, p a prime and n a positive integer. The design of (1.1) corresponding to $s = 2$ was constructed by Bose and Shimamoto [**4**] by another method. All other designs of (1.1) are new.

Bose [**1**] and Bose and Nair [**3**] have used finite geometries and Galois fields in the construction of incomplete block designs. Once the geometrical configuration is determined, the proper assignment of the treatments to the blocks of design (1.1) may be determined by the geometrical method which is illustrated for the design corresponding to $s = 3$.

2. A "working" definition of partially balanced incomplete block designs with two associate classes. Partially balanced incomplete block designs (P.B.I.B. designs) with m associate classes ($m \geqq 1$) were introduced by Bose and Nair [**3**]. Later Nair and Rao [**5**] broadened the definition of P.B.I.B. designs. Recently Bose and Shimamoto [**4**] rephrased the definition in order to stress the fact that the relations of association between the treatments are determined only by the parameters n_i and p^i_{jk} ($i, j, k = 1, 2, \cdots, m$). For the special case of P.B.I.B. designs with two associate classes ($m = 2$), Bose and Clatworthy [**2**] have proposed a new definition which to-

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gether with two theorems are equivalent to the Bose and Shimamoto definition. Use will be made in the present paper of the definition and theorems of [2] which are stated below.

DEFINITION. A P.B.I.B. design with two associate classes is an arrangement of v treatments in b blocks such that:

(1) Each of the v treatments occurs in r blocks. Each block is of size k (i.e., contains k treatments), and no treatment appears more than once in any block.

(2) There exists a relationship of association between any pair of the v treatments satisfying the following conditions:

(a) Any two treatments are either first or second associates.

(b) Each treatment has n_1 first and n_2 second associates.

(c) For any pair of the v treatments which are ith associates, the number of treatments common to the first associates of the first and the first associates of the second is p^i_{11} $(i=1, 2)$, and this number is independent of the pair of treatments with which we start.

(3) Any pair of treatments which are ith associates occur together in exactly λ_i blocks $(i=1, 2)$.

THEOREM 1. *For every pair of first associates among the v treatments of a P.B.I.B. design with two associate classes, the numbers p^1_{12}, p^1_{21}, and p^1_{22} are constants, and $p^1_{12}=p^1_{21}$.*

THEOREM 2. *For every pair of second associates among the v treatments of a P.B.I.B. design with two associate classes, the numbers p^2_{12}, p^2_{21}, and p^2_{22} are constants, and $p^2_{12}=p^2_{21}$.*

In proving Theorems 1 and 2 the following useful relations were established:

$$p^1_{12} = p^1_{21} = n_1 - p^1_{11} - 1, \tag{2.1}$$

$$p^1_{22} = n_2 - n_1 + p^1_{11} + 1, \tag{2.2}$$

$$p^2_{12} = p^2_{21} = n_1 - p^2_{11}, \tag{2.3}$$

$$p^2_{22} = n_2 - n_1 + p^2_{11} - 1. \tag{2.4}$$

The four preceding expressions are special cases of general expressions derived in [3] where the following relations were also established:

$$vr = bk, \tag{2.5}$$

$$v = n_1 + n_2 + 1, \tag{2.6}$$

$$\lambda_1 n_1 + \lambda_2 n_2 = r(k - 1), \tag{2.7}$$

$$n_1 \overset{1}{p}_{12} = n_2 \overset{2}{p}_{11}, \qquad n_1 \overset{1}{p}_{22} = n_2 \overset{2}{p}_{12}. \tag{2.8}$$

Furthermore, it was shown that if values are assigned to the parameters of the first kind (v, b, r, k, λ_1, λ_2, n_1, and n_2) satisfying (2.5), (2.6), and (2.7), then only one of the parameters of the second kind (p^i_{jk}, $i, j, k = 1, 2$) is independent.

The parameters of the second kind are exhibited as elements of two symmetric matrices,

$$P_1 = \begin{pmatrix} \overset{1}{p}_{11} & \overset{1}{p}_{12} \\ \overset{1}{p}_{21} & \overset{1}{p}_{22} \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} \overset{2}{p}_{11} & \overset{2}{p}_{12} \\ \overset{2}{p}_{21} & \overset{2}{p}_{22} \end{pmatrix}. \tag{2.9}$$

3. **The geometrical configuration.** Consider the finite projective geometry $PG(3, s)$ which has $(s^4-1)/(s-1)$ points, $(s^2+1)(s^2+s+1)$ lines, and $(s^4-1)/(s-1)$ planes [1]. In this geometry the number of points contained in a line is $s+1$ and the number of points in a plane is s^2+s+1. Through any point there pass s^2+s+1 lines and the same number of planes. One line and $s+1$ planes pass through every pair of points in $PG(3, s)$.

The $(s^4-1)/(s-1)$ planes of $PG(3, s)$ in which we have an interest are chosen such that they are in (1-1) correspondence with the points of $PG(3, s)$. Let

$$U_i = \sum_{j=1}^{4} a_{ij}x_j \qquad (i = 1, 2, 3, 4) \tag{3.1}$$

where a_{ij} are elements of the Galois field $GF(s)$, $s = p^n$, and the matrix (a_{ij}) is skew-symmetric. To the point $\theta = (l_1, l_2, l_3, l_4)$ of $PG(3, s)$ let there correspond the plane π_θ whose equation is

$$\sum_{i=1}^{4} l_i U_i = 0. \tag{3.2}$$

We shall now establish the following

LEMMA 3.1. (i) *The plane π_θ passes through the point θ.*

(ii) *If the plane π_θ passes through the point ϕ, then the plane π_ϕ passes through point θ.*

By (3.1) and (3.2) the equation of the plane π_θ corresponding to point $\theta = (l_1, l_2, l_3, l_4)$ can be expressed in the form

$$\sum_{i,j=1}^{4} a_{ij} l_i x_j = 0. \tag{3.3}$$

Using the fact that the matrix (a_{ij}) is skew-symmetric, (3.3) may be written as

$$\begin{aligned}(3.4)\qquad & a_{12}(l_1x_2 - l_2x_1) + a_{13}(l_1x_3 - l_3x_1) + a_{14}(l_1x_4 - l_4x_1) \\ & + a_{23}(l_2x_3 - l_3x_2) + a_{24}(l_2x_4 - l_4x_2) + a_{34}(l_3x_4 - l_4x_3) = 0.\end{aligned}$$

Obviously equation (3.4) is satisfied when

$$(3.5)\qquad (x_1, x_2, x_3, x_4) = (l_1, l_2, l_3, l_4) = \theta.$$

Hence point θ lies in plane π_θ.

Let point ϕ be denoted by (m_1, m_2, m_3, m_4). Since plane π_θ corresponding to point θ passes through point ϕ by hypothesis, we have from (3.3)

$$(3.6)\qquad \sum_{i,j=1}^{4} a_{ij}l_im_j = 0.$$

The plane π_ϕ corresponding to point ϕ is

$$(3.7)\qquad \sum_{i=1}^{4} m_iU_i = 0$$

or

$$(3.8)\qquad \sum_{i,j=1}^{4} a_{ij}m_ix_j = 0.$$

For plane π_ϕ to pass through point θ it is necessary that the coordinates of θ satisfy (3.8), i.e.,

$$(3.9)\qquad \sum_{i,j=1}^{4} a_{ij}m_il_j = 0.$$

Since matrix (a_{ij}) is skew-symmetric, (3.9) becomes

$$(3.10)\qquad -\sum_{i,j=1}^{4} a_{ji}m_il_j = 0,$$

which is identical to (3.6). Therefore, if plane π_θ passes through point ϕ, then plane π_ϕ passes through point θ.

4. Interpretation of the geometrical configuration as design (1.1). We now identify the $(s^4-1)/(s-1)$ points of $PG(3, s)$ with the $(s^4-1)/(s-1)$ treatments of design (1.1) such that a point of $PG(3, s)$ corresponds to one and only one treatment of the design.

First, we establish the relation of association for design (1.1). We shall say that the point (or treatment) ϕ is a first associate of point

θ if the plane π_θ corresponding to θ passes through point ϕ. Let ϕ be a first associate of θ. Then by Lemma 3.1, plane π_ϕ passes through point θ; hence, point θ is by definition a first associate of point ϕ. Thus the association relation is reflexive as it should be. By definition, any point α which is not a first associate of θ is its second associate.

We shall now prove that condition (c) of (2) of the definition of a P.B.I.B. design with two associate classes is satisfied.

(i) Since there are s^2+s points lying in π_θ other than θ itself, the number of first associates of θ is

$$n_1 = s^2 + s. \tag{4.1}$$

Since the total number of points in $PG(3, s)$ is s^3+s^2+s+1, the number of second associates of θ is

$$n_2 = s^3. \tag{4.2}$$

(ii) Let points θ and ϕ be first associates. Points which are first associates of both θ and ϕ must lie in the line of intersection of planes π_θ and π_ϕ passing through θ and ϕ respectively. This line passes through both θ and ϕ. It contains $s-1$ points other than θ and ϕ. Hence the number of points common to the first associates of θ and ϕ is

$$\overset{1}{p}_{11} = s - 1. \tag{4.3}$$

Next, let points θ and ϕ be second associates. The points which are first associates of both θ and ϕ must lie on the line of intersection of the planes π_θ and π_ϕ as before. However, this line no longer passes through points θ and ϕ. Since a line contains $s+1$ points, the number of treatments common to the first associates of θ and ϕ is

$$\overset{2}{p}_{11} = s + 1. \tag{4.4}$$

It now follows from Theorems 1 and 2 and expressions (2.1) through (2.4) that

$$\overset{1}{p}_{12} = \overset{1}{p}_{21} = n_1 - \overset{1}{p}_{11} - 1 = s^2, \tag{4.5}$$

$$\overset{1}{p}_{22} = n_2 - n_1 + \overset{1}{p}_{11} + 1 = s^2(s - 1), \tag{4.6}$$

$$\overset{2}{p}_{12} = \overset{2}{p}_{21} = n_1 - \overset{2}{p}_{11} = s^2 - 1, \tag{4.7}$$

$$\overset{2}{p}_{22} = n_2 - n_1 + \overset{2}{p}_{11} - 1 = s^2(s - 1). \tag{4.8}$$

It remains to show that the conditions on parameters b, r, k, λ_1, and λ_2 of design (1.1) are satisfied. Let l_θ be any line passing through

θ and lying in the plane π_θ. The set of all points contained in line l_θ is taken to form one block of our design. Since there are $s+1$ lines passing through θ and lying in π_θ, there are $s+1$ blocks containing treatment θ. Hence

$$r = k = s + 1. \tag{4.9}$$

The total number of points in $PG(3, s)$ is $(s^4-1)/(s-1)$. Hence the total number of lines like l_θ is $(s^4-1)/(s-1)$ since through each point there pass $s+1$ lines like l_θ, and on each line there lie $s+1$ points. Thus the number of blocks is

$$b = (s^4 - 1)/(s - 1). \tag{4.10}$$

If θ and ϕ are first associates, then the line joining θ and ϕ is a block. Hence any two first associates occur together in exactly one block, giving

$$\lambda_1 = 1. \tag{4.11}$$

If θ and ϕ are second associates, neither of the corresponding planes π_θ and π_ϕ contains both θ and ϕ. Hence the lines l_θ passing through θ never pass through ϕ and, therefore, the blocks containing θ cannot contain ϕ. Thus

$$\lambda_2 = 0. \tag{4.12}$$

This completes the proof that the system of blocks defined above satisfies design (1.1).

5. **Construction of the design with** $s=3$. The geometrical method of construction of design (1.1) will be illustrated by considering the design corresponding to $s=3$. This design has parameters

$$\begin{aligned} v = b = 40,\ \lambda_1 = 1,\ n_1 = 12,\ P_1 &= \begin{pmatrix} 2 & 9 \\ 9 & 18 \end{pmatrix}, \\ r = k = 4,\ \ \lambda_2 = 0,\ n_2 = 27,\ P_2 &= \begin{pmatrix} 4 & 8 \\ 8 & 18 \end{pmatrix}. \end{aligned} \tag{5.1}$$

We use $PG(3, 3)$ and the Galois field $GF(3)$ whose $s=3$ elements are 0, 1, and 2. Let the skew-symmetric matrix be

$$(a_{ij}) = \left[\begin{array}{cc:cc} 0 & 1 & & \\ 2 & 0 & \multicolumn{2}{c}{0} \\ \hdashline & & 0 & 2 \\ \multicolumn{2}{c:}{0} & 1 & 0 \end{array}\right]. \tag{5.2}$$

The plane of $PG(3, 3)$ corresponding to point $\theta = (l_1, l_2, l_3, l_4)$ is

$$l_1x_2 + 2l_2x_1 + 2l_3x_4 + l_4x_3 = 0. \tag{5.3}$$

In order to have a one-to-one correspondence between sets of coordinates with elements in $GF(3)$ and the 40 points of $PG(3, 3)$, we shall restrict ourselves to the set of coordinates in which the left-most non-zero digit is unity. In order to express the blocks of the design in terms familiar to the experimentalist, we set up a correspondence between the points of $PG(3, 3)$ and the integers $1, 2, 3, \cdots, 40$ according to the following rule: Let (l_1, l_2, l_3, l_4) correspond to

$$l_1s^3 + l_2s^2 + l_3s + l_4 - \alpha$$

where the number α is defined by

$$\begin{aligned} \alpha &= s^2 + s + 1 && \text{if } l_1 = 1, \\ \alpha &= s + 1 && \text{if } l_1 = 0 \text{ and } l_2 = 1, \\ \alpha &= 1 && \text{if } l_1 = l_2 = 0 \text{ and } l_3 = 1, \\ \alpha &= 0 && \text{if } l_1 = l_2 = l_3 = 0 \text{ and } l_4 = 1. \end{aligned}$$

The solution of design (5.1) is exhibited below in tabular form. It will be noted that only 13 of the 40 planes are exhibited in the solution. The points of the other 27 planes lie in the 13 planes indicated in column (2).

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SOLUTION OF DESIGN (5.1)

(1) Point	(2) Plane Corresp. to Point	(3) Points Lying in Lines (blocks) of Plane				(4) Blocks of (5.1)			
0001	$x_3=0$	0001	0100	0101	0102	1	5	6	7
		0001	1000	1001	1002	1	14	15	16
		0001	1100	1101	1102	1	23	24	25
		0001	1200	1201	1202	1	32	33	34
0100	$x_1=0$	0100	0010	0110	0120	5	2	8	11
		0100	0011	0111	0122	5	3	9	13
		0100	0012	0112	0121	5	4	10	12
0101	$2x_1+x_3=0$	0101	1010	1111	1212	6	17	27	37
		0101	1011	1112	1210	6	18	28	35
		0101	1012	1110	1211	6	19	26	36

SLUTIOON OF DESIGN (5.1) (continued)

(1) Point	(2) Plane Corresp. to Point	(3) Points Lying in Lines (blocks) of Plane				(4) Blocks of (5.1)			
0102	$x_1+x_3=0$	0102	1020	1122	1221	7	20	31	39
		0102	1021	1120	1222	7	21	29	40
		0102	1022	1121	1220	7	22	30	38
1000	$x_2=0$	1000	0010	1010	1020	14	2	17	20
		1000	0011	1011	1022	14	3	18	22
		1000	0012	1012	1021	14	4	19	21
1001	$x_2+x_3=0$	1001	0120	1121	1211	15	11	30	36
		1001	0121	1122	1210	15	12	31	35
		1001	0122	1120	1212	15	13	29	37
1002	$x_2+2x_3=0$	1002	0110	1112	1222	16	8	28	40
		1002	0111	1110	1221	16	9	26	39
		1002	0112	1111	1220	16	10	27	38
1100	$x_2+2x_1=0$	1100	0010	1110	1120	23	2	26	29
		1100	0011	1111	1122	23	3	27	31
		1100	0012	1112	1121	23	4	28	30
1101	$2x_1+x_2+x_3=0$	1101	0120	1011	1221	24	11	18	39
		1101	0121	1010	1222	24	12	17	40
		1101	0122	1012	1220	24	13	19	38
1102	$2x_1+x_2+2x_3=0$	1102	0110	1022	1212	25	8	22	37
		1102	0111	1021	1210	25	9	21	35
		1102	0112	1020	1211	25	10	20	36
1200	$x_1+x_2=0$	1200	0010	1210	1220	32	2	35	38
		1200	0011	1211	1222	32	3	36	40
		1200	0012	1212	1221	32	4	37	39
1201	$x_1+x_2+x_3=0$	1201	0120	1021	1111	33	11	21	27
		1201	0121	1022	1110	33	12	22	26
		1201	0122	1020	1112	33	13	20	28
1202	$x_1+x_2+2x_3=0$	1202	0110	1012	1122	34	8	19	31
		1202	0111	1010	1121	34	9	17	30
		1202	0112	1011	1120	34	10	18	29

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SOME CONDITIONS FOR UNIFORM CONVERGENCE OF INTEGRALS

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Let m be a sigma-finite measure on a space R, let T be an abstract index set, and for each $t \in T$ and $n=1, 2, \cdots$, let $f_n(x, t)$, $f(x, t)$ be functions measurable in $x \in R$. It is of some interest to determine conditions for statements of the form

$$(1) \qquad \int_R |f_n(x, t) - f(x, t)| \, dm \to 0 \qquad \text{uniformly in } t,$$

where we use the conventions that if a limit is taken as $n \to \infty$, or if t varies in T, then these facts will not be explicitly stated. Graves [1, p. 241, Theorem 22] states conditions for (1) to hold under the assumption that:

(2) at each $x \in R$, except for x in a null set which is independent of t, $f_n(x, t)$ converges to $f(x, t)$ uniformly in t.

The mode of convergence defined by (2) is much too restrictive. In this paper we define modes of uniform convergence of a family of sequences of measurable functions which are much less restrictive than (2) and under which (1) holds. Situations of this more general kind were encountered by the author in [2], and the theorems proved below are there applied.

Before introducing our modes of convergence, let us recall that, given measurable functions f, f_n $(n=1, 2, \cdots)$, Egorov's theorem states that $f_n(x) \to f(x)$ almost everywhere (that is, $m\{x: f_n(x)$ does not converge to $f(x)\}=0$) if, and only if, for every $\epsilon>0$ and every

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