THE INTEGRAL OF A FUNCTION WITH RESPECT TO A FUNCTION

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1. **Introduction.** For the Stieltjes integral $\int_a^b u(x)dv(x)$ to exist, *it is necessary that if x is a number in the interval $[a, b]$, then u or v is continuous at x.* In this paper we give a simple definition (based on the trapezoidal rule) of a more general integral such that

(i) *Both* $\int_a^b u(x)dv(x)$ and $\int_a^b v(x)du(x)$ exist if, for example, u is *any* function which has only discontinuities of the first kind and v is ***any*** function of bounded variation,

(ii) integration by parts and other desirable properties of the Stieltjes integral are retained (see Theorem 2.1), and

(iii) the formula $\int_a^b u(x)d\left[\int_a^x v(t)dw(t)\right] = \int_a^b u(x)v(x)dw(x)$ holds under rather general conditions (see Theorem 4.2).

2. **Definition and fundamental properties of the integral.** In this paper we consider only functions which are defined over the set of all real numbers; i.e., the statement that f is a function means that if x is a real number, then $f(x)$ is a number.

If $[a, b]$ is an interval, then the statement that D is a subdivision of $[a, b]$ means that D is a set, $x_0, x_1, \cdots, x_n$, of real numbers such that $a = x_0 < x_1 < \cdots < x_n = b$; the statement that D_1 is a refinement of D means that D_1 is a subdivision of $[a, b]$ and that D is a subset of D_1. If $[a, b]$ is an interval, u is a function, v is a function, and D is a subdivision $x_0, x_1, \cdots, x_n$ of $[a, b]$, then we denote by $S_D(u, v)$ the sum

$$\sum_{i=0}^{n-1} \frac{1}{2} \left[u(x_i) + u(x_{i+1})\right]\left[v(x_{i+1}) - v(x_i)\right].$$

DEFINITION 2.1. Suppose that $[a, b]$ is an interval, u is a function, and v is a function. Then

(i) the statement that u is integrable with respect to v in $[a, b]$ means that if ϵ is a positive number, then there is a subdivision D of $[a, b]$ such that if D_1 is a refinement of D then $|S_D(u, v) - S_{D_1}(u, v)| < \epsilon$, and

(ii) the statement that $\int_a^b u(x)dv(x) = I$ means that I is a number and that if ϵ is a positive number, then there is a subdivision D of $[a, b]$ such that if D_1 is a refinement of D, then $|S_{D_1}(u, v) - I| < \epsilon$; moreover, if $\int_a^b u(x)dv(x) = I$, then $\int_b^a u(x)dv(x) = -\int_a^b u(x)dv(x)$; and

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$\int_c^c u(x)dv(x) = 0$ if c is a real number.

LEMMA 2.1a. *Suppose that* $[a, b]$ *is an interval,* u *is a function, and* v *is a function. Then each of the following three statements implies the other two*:

(i) u *is integrable with respect to* v *in* $[a, b]$;

(ii) v *is integrable with respect to* u *in* $[a, b]$;

(iii) *there is a number* I *such that* $\int_a^b u(x)dv(x) = I$.

PROOF. From the identity $S_D(u, v) + S_D(v, u) = u(b)v(b) - u(a)v(a)$ it follows that (i) implies (ii) and vice versa. Moreover, (iii) obviously implies (i). Suppose now that (i) is true and that ϵ is a positive number. Let D denote a subdivision of $[a, b]$ such that if D_1 is a refinement of D then $|S_D(u, v) - S_{D_1}(u, v)| < \epsilon$, and let E denote a subdivision of $[a, b]$ such that if E_1 is a refinement of E then $|S_E(u, v) - S_{E_1}(u, v)| < \epsilon$. Let F denote a refinement of D which is also a refinement of E. Then $|S_D(u, v) - S_E(u, v)| \leqq |S_D(u, v) - S_F(u, v)| + |S_E(u, v) - S_F(u, v)| < 2\epsilon$. By Cauchy's convergence criterion it follows that (ii) is true. This completes the proof.

REMARK 2.1. Since $S_D(u, v)$ is half of the sum of two approximating sums for a Stieltjes integral, it follows that if the Stieltjes integral exists, then u is integrable with respect to v in $[a, b]$, and that $\int_a^b u(x)dv(x)$ is equal to the Stieltjes integral.

REMARK 2.2. Suppose that if x, x', x'' are in $[a, b]$ then

$$\begin{vmatrix} u(x) & v(x) & 1 \\ u(x') & v(x') & 1 \\ u(x'') & v(x'') & 1 \end{vmatrix} = 0.$$

Then u is integrable with respect to v in $[a, b]$, and $\int_a^b u(x)dv(x) = 2^{-1}[u(a) + u(b)][v(b) - v(a)]$. In particular, if $[a, b]$ is an interval and u is a function, then u is integrable with respect to u in $[a, b]$.

REMARK 2.3. Suppose that $u(x) = 0$ if $x < 1/2$ and $u(x) = 2$ if $x \geqq 1/2$. Then $\int_0^1 u(x)du(x) = 2$. The corresponding Stieltjes integral does not exist. The Lebesgue-Stieltjes integral has the value 4, so that integration by parts does not hold.

THEOREM 2.1. *If* $[a, b]$ *is an interval,* u *is a function, and* v *is a function such that* u *is integrable with respect to* v *in* $[a, b]$, *then each of the following statements is true.*

(i) *If* k *is a number, then* u *is integrable with respect to* $v + k$ *in* $[a, b]$, *and* $\int_a^b u(x)d[v(x) + k] = \int_a^b u(x)dv(x)$.

(ii) If $a < c < b$, *then* u *is integrable with respect to* v *in* $[a, c]$ *and in* $[c, b]$, *and* $\int_a^c u(x)dv(x) + \int_a^b u(x)dv(x) = \int_c^b u(x)dv(x)$.

(iii) *If* u_1 *is a function which is integrable with respect to* v *in* $[a, b]$, *then* $u+u_1$ *is integrable with respect to* v *in* $[a, b]$, *and* $\int_a^b[u(x)+u_1(x)]dv(x)=\int_a^b u(x)dv(x)+\int_a^b u_1(x)dv(x)$.

(iv) *If* k *is a number, then* ku *is integrable with respect to* v *in* $[a, b]$, *and* $\int_a^b ku(x)dv(x)=k\int_a^b u(x)dv(x)$.

(v) $\int_a^b u(x)dv(x)=u(b)v(b)-u(a)v(a)-\int_a^b v(x)du(x)$.

(vi) *If* v *is of bounded variation in* $[a, b]$, M *is a positive number, and* $|u(x)|<M$ *for each number* x *in* $[a, b]$, *then* $|\int_a^b u(x)dv(x)| \leqq MV_a^b(v)$.

Proof is omitted, since the theorem follows readily from Definition 2.1 and Lemma 2.1a. In connection with the proof of (ii) of the theorem, we make the following observation. Suppose that D is a subdivision of $[a, b]$ such that c is in D, and that D_1 is a refinement of D such that if x is in D_1, then x is in D or x is in $[a, c]$; then the difference $S_D(u, v)-S_{D_1}(u, v)$ is equal to $S_E(u, v)-S_{E_1}(u, v)$, where E is a subdivision of $[a, c]$ and E_1 is a refinement of E.

3. Integral of a function with respect to a step-function.

The statement that u is a step-function means that u is a function such that if $[a, b]$ is an interval, then there are a subdivision, $x_0, x_1, \cdots, x_n$, of $[a, b]$ and a sequence, $c_0, c_1, \cdots, c_{n-1}$, of numbers such that if i is one of the integers $0, 1, \cdots, n-1$, and $x_i<x<x_{i+1}$, then $u(x)=c_i$.

We denote by A the set such that u is in A if and only if u is a function such that if x is a real number, then the limits $u(x-)$ and $u(x+)$ exist.

THEOREM 3.1. *If* $[a, b]$ *is an interval,* u *is in* A, *and* v *is a step-function, then* u *is integrable with respect to* v *in* $[a, b]$, *and*

$$\begin{aligned}\int_a^b u(x)dv(x) &= \sum_{a<x\leqq b}\frac{1}{2}[u(x-)+u(x)][v(x)-v(x-)] \\ &\quad+\sum_{a\leqq x<b}\frac{1}{2}[u(x)+u(x+)][v(x+)-v(x)].\end{aligned}\tag{3.1}$$

PROOF. Let V denote the total variation of v in $[a, b]$. The case in which $V=0$ is trivial; so we suppose that $V>0$. Let $x_0, x_3, \cdots, x_{3n}$ denote a subdivision of $[a, b]$ such that if x is in $[a, b]$ and v is not continuous at x, then x is one of the numbers $x_0, x_3, \cdots, x_{3n}$. Suppose that ϵ is a positive number. If i is one of the integers $0, 1, \cdots, n-1$, then there are a number x_{3i+1} and a number x_{3i+2} such that $x_{3i}<x_{3i+1}<x_{3i+2}<x_{3i+3}$, and $|u(x)-u(x_{3i}+)|<\epsilon$ if $x_{3i}<x\leqq x_{3i+1}$, and $|u(x)-u(x_{3i+3}-)|<\epsilon$ if $x_{3i+2}\leqq x<x_{3i+3}$. Hence if I denotes the right-hand

member of (3.1), D denotes the subdivision $x_0, x_1, \cdots, x_{3n}$, and D_1 is a refinement of D, then $|S_{D_1}(u, v) - I| < \epsilon V$. This completes the proof.

REMARK 3.1. A function u belongs to the set A if and only if it is true that if v is a step-function and $[a, b]$ is an interval, then u is integrable with respect to v in $[a, b]$.

REMARK 3.2. Suppose that u is a distribution function (i.e., u is a nondecreasing real-valued function everywhere continuous from the right, and $u(x) \to 0$ as $x \to -\infty$, and $u(x) \to 1$ as $x \to +\infty$), and v is a distribution function which is a step-function. If for each real number x it is true that $f(x) = \int_{-\infty}^{+\infty} u(x-t)dv(t)$, then f is a distribution function. Thus we see that Definition 2.1 has an almost immediate application to the theory of probability and statistics.

4. Integral of a function with respect to a function of bounded variation. We begin by proving two lemmas.

LEMMA 4.1a. *Suppose that $[a, b]$ is an interval, v is a function which is of bounded variation in $[a, b]$, and $u_1, u_2, u_3, \cdots$ is a sequence of functions such that*

(i) *If n is a positive integer, then u_n is integrable with respect to v in $[a, b]$, and*

(ii) *there is a function u such that $u_1, u_2, u_3, \cdots$ converges to u uniformly in $[a, b]$. Then u is integrable with respect to v in $[a, b]$, and $\int_a^b u_n(x)dv(x) \to \int_a^b u(x)dv(x)$ as $n \to \infty$.*

PROOF. Let V denote the total variation of v in $[a, b]$. The case in which $V=0$ is trivial. Suppose that $V>0$ and that ϵ is a positive number. Let N denote a positive number such that if n is an integer greater than N and x is in $[a, b]$, then $|u(x) - u_n(x)| < \epsilon$. If m is an integer greater than N and n is an integer greater than N, then by (iii), (iv), and (vi) of Theorem 2.1, $|\int_a^b u_m(x)dv(x) - \int_a^b u_n(x)dv(x)| < 2\epsilon V$. Moreover, there is a number I such that $\int_a^b u_n(x)dv(x) \to I$ as $n \to \infty$; and if n is an integer greater than N, then $|\int_a^b u_n(x)dv(x) - I| < 2\epsilon V$. Now suppose that n is an integer greater than N and that D is a subdivision of $[a, b]$ such that if D_1 is a refinement of D then $|\int_a^b u_n(x)dv(x) - S_{D_1}(u_n, v)| < \epsilon V$. If D_1 is a refinement of D, then $|I - S_{D_1}(u, v)| \leqq |I - \int_a^b u_n(x)dv(x)| + |\int_a^b u_n(x)dv(x) - S_{D_1}(u_n, v)| + |S_{D_1}(u_n, v) - S_{D_1}(u, v)| < 4\epsilon V$. This completes the proof.

REMARK 4.1. The following result can be proved by a similar argument. Suppose that $[a, b]$ is an interval, u is a function which is bounded in $[a, b]$, and $v_1, v_2, v_3, \cdots$, is a sequence of functions such that

(i) if n is a positive integer, then u is integrable with respect to v_n in $[a, b]$, and

(ii) there is a function v such that $V_a^b(v-v_n)\to 0$ as $n\to\infty$. Then u is integrable with respect to v in $[a, b]$, and $\int_a^b u(x)dv_n(x)\to\int_a^b u(x)dv(x)$ as $n\to\infty$.

The second lemma concerns the set A introduced in §3.

LEMMA 4.1b. *For u to be in A, it is necessary and sufficient that u be a function such that if $[a, b]$ is an interval and ϵ is a positive number, then there is a step-function u_1 such that if x is in $[a, b]$ then $|u_1(x)-u(x)|<\epsilon$.*

PROOF. A. Suppose that u is in A, $[a, b]$ is an interval, and ϵ is a positive number. If there is in $[a, b]$ a number x such that $|u(x-)-u(x)|+|u(x)-u(x+)|\geqq\epsilon/2$, let σ denote the set of all such numbers x in $[a, b]$. If σ is an infinite set, then it has a limit-point, x_1, in $[a, b]$, and one of the limits $u(x_1-)$, $u(x_1+)$ does not exist, contrary to hypothesis; hence σ is a finite set. Let v_0 denote a step-function such that (1) if x is in $[a, b]$ but not in σ, then v_0 is continuous at x, and (2) if x is in σ, then $v_0(x-)-v_0(x)=u(x-)-u(x)$ and $v_0(x)-v_0(x+)=u(x)-u(x+)$. Suppose that $v=u-v_0$. Then v is in A; and if x is in $[a, b]$, then $|v(x-)-v(x)|+|v(x)-v(x+)|<\epsilon/2$. Let s denote the increasing sequence $x_0, x_1, x_2, \cdots$, constructed as follows. The number x_0 is a; and x_1 is the largest number in $[a, b]$ such that if $x_0<x<x_1$ then $|v(x_0+)-v(x)|<\epsilon$. If $x_1<b$, then x_2 is the largest number in $[a, b]$ such that if $x_1<x<x_2$ then $|v(x_1+)-v(x)|<\epsilon$. If $x_2<b$, then x_3 is the largest number in $[a, b]$ such that if $x_2<x<x_3$ then $|v(x_2+)-v(x)|<\epsilon$; and so on. If there are infinitely many terms in s, then s converges to a number x' which is in $[a, b]$, and the limit $v(x'-)$ does not exist, contrary to hypothesis. Hence there is a positive integer n such that $x_n=b$. Let v_1 denote a step-function such that $v_1(b)=v(b)$ and such that if i is one of the integers $0, 1, \cdots, n-1$, then $v_1(x_i)=v(x_i)$ and $v_1(x)=v(x_i+)$ if $x_i<x<x_{i+1}$. Suppose that u_1 is a step-function such that $u_1=v_0+v_1$. If x is in $[a, b]$, then $|u(x)-u_1(x)|=|u(x)-v_0(x)-v_1(x)|=|v(x)-v_1(x)|<\epsilon$.

B. Suppose that u is a function such that if $[a, b]$ is an interval and ϵ is a positive number, then there is a step-function u_1 such that if x is in $[a, b]$ then $|u(x)-u_1(x)|<\epsilon$. Suppose that $[a, b]$ is an interval, ϵ is a positive number, and u_1 is a step-function such that if x is in $[a, b+\epsilon]$, then $|u(x)-u_1(x)|<\epsilon$. Let x denote a number in $[a, b]$. There is a positive number δ less than ϵ such that if $x<x'<x''<x+\delta$, then $|u_1(x')-u_1(x'')|<\epsilon$, so that $|u(x')-u(x'')|<3\epsilon$; hence the limit $u(x+)$ exists. Similarly, the limit $u(x-)$ exists, and u is in the set

A. This completes the proof.

THEOREM 4.1. *If $[a, b]$ is an interval, u is a function in the set A, and v is a function which is of bounded variation in $[a, b]$, then u is integrable with respect to v in $[a, b]$.*

PROOF. Suppose that $u_1, u_2, u_3, \cdots$ is a sequence of step-functions which converges to u uniformly in $[a, b]$. Now v is of bounded variation in $[a, b]$; so if $a < x \leqq b$, then $v(x-)$ exists, and if $a \leqq x < b$ then $v(x+)$ exists. Hence if n is a positive integer, then v is integrable with respect to u_n in $[a, b]$, and the number $\int_a^b u_n(x)dv(x)$ can be found by (3.1) and integration by parts. By Lemma 4.1a, it follows that u is integrable with respect to v in $[a, b]$ and that $\int_a^b u_n(x)dv(x) \to \int_a^b u(x)dv(x)$ as $n \to \infty$. This completes the proof.

REMARK 4.2. A function u belongs to the set A if and only if it is true that if v is a function which is of bounded variation in an interval then u is integrable with respect to v in that interval.

LEMMA 4.2a. *Suppose that $[a, b]$ is an interval, u is a function in A, and $v_1, v_2, v_3, \cdots$ is a sequence of functions such that*

(i) *there is a positive number V such that $V_a^b(v_n) < V$ for $n = 1, 2, 3, \cdots$, and*

(ii) *there is a function v such that $v_1, v_2, v_3, \cdots$ converges to v uniformly in $[a, b]$. Then $\int_a^x u(t)dv_n(t) \to \int_a^x u(t)dv(t)$ uniformly in $[a, b]$ as $n \to \infty$.*

PROOF. A. We show first that v is of bounded variation in $[a, b]$ and that $V_a^b(v) \leqq V$. Suppose that this is not the case. Then there are a positive number ϵ and a subdivision $x_0, x_1, \cdots, x_p$ of $[a, b]$ such that $\sum_{i=0}^{p-1} |v(x_{i+1}) - v(x_i)| > V + 2\epsilon$; and by (ii) of the hypothesis there is a number N such that if n is an integer greater than N then $\sum_{i=0}^{p-1} |v_n(x_{i+1}) - v_n(x_i)| > \sum_{i=0}^{p-1} |v(x_{i+1}) - v(x_i)| - \epsilon > V + \epsilon$, which is contrary to (i) of the hypothesis.

B. Suppose that $a < x \leqq b$. Let $u_1, u_2, u_3, \cdots$, denote a sequence of step-functions which converges to u uniformly in $[a, b]$. We use the following notation: if m is a positive integer and n is a positive inger, then $I_{m,n} = \int_a^x u_m(t)dv_n(t)$, $I_m = \int_a^x u_m(t)dv(t)$, $J_n = \int_a^x u(t)dv_n(t)$, and $I = \int_a^x u(t)dv(t)$. Suppose that ϵ is a positive number. Let m denote a positive integer such that if t is in $[a, b]$, then $|u_m(t) - u(t)| < \epsilon$, and let M denote a positive number such that if t is in $[a, b]$, then $2|u_m(t)| + V_a^b(u_m) \leqq M$. Let n denote a positive integer such that if t is in $[a, b]$ then $|v_n(t) - v(t)| M < \epsilon V$. Then

$$|J_n - I| \leqq |J_n - I_{m,n}| + |I_{m,n} - I_m| + |I_m - I|,$$

where $|J_n - I_{m,n}| = |\int_a^x [u(t) - u_m(t)]\, dv_n(t)| < \epsilon V$, $|I_{m,n} - I_m| = |\int_a^x u_m(t) d[v_n(t) - v(t)]| = |u_m(t)[v_n(t) - v(t)]|_a^x - \int_a^x [v_n(t) - v(t)] du_m(t)| < \epsilon V$, and $|I_m - I| = |\int_a^x [u_m(t) - u(t)] dv(t)| < \epsilon V$. Hence if $a < x \leqq b$, then $|J_n - I| < 3\epsilon V$. If $x = a$, then $\int_a^x u(t) dv_n(t) = \int_a^x u(t) dv(t) = 0$. This completes the proof.

THEOREM 4.2. *Suppose that $[a, b]$ is an interval, u is in A, v is in A, and w is a function of bounded variation in $[a, b]$. Suppose further that (1) if $a < x \leqq b$, then one of u, v, w is continuous from the left at x, and (2) if $a \leqq x < b$, then one of u, v, w is continuous from the right at x. Then $\int_a^b u(x) d[\int_a^x v(t) dw(t)] = \int_a^b u(x) v(x) dw(x)$.*

PROOF. For each real number x in $[a, b]$, let $f(x)$ denote the number $\int_a^x v(t) dw(t)$.

Case 1. Suppose that w is a step-function. Then f is a step-function; moreover, if $a < x \leqq b$, then $f(x) - f(x-) = 2^{-1}[v(x-) + v(x)][w(x) - w(x-)]$, and if $a \leqq x < b$, then $f(x+) - f(x) = 2^{-1}[v(x) + v(x+)][w(x+) - w(x)]$. From these facts, the algebraic identity $2^{-1}(u_1 + u_2) \cdot 2^{-1}(v_1 + v_2)(w_2 - w_1) = 2^{-1}(u_1 v_1 + u_2 v_2)(w_2 - w_1) - 4^{-1}(u_2 - u_1)(v_2 - v_1) \cdot (w_2 - w_1)$, and (3.1), it follows at once that

$$\int_a^b u(x) df(x) = \int_a^b u(x) v(x) dw(x).$$

Case 2. Suppose that w is not a step-function. If there is a number x such that $a < x \leqq b$ and one of u, v, w is not continuous from the left at x, let L denote the set of all such numbers x. If there is a number x such that $a \leqq x < b$ and one of u, v, w is not continuous from the right at x, let R denote the set of all such numbers x. From the proof of Lemma 4.1b and the fact that w is of bounded variation in $[a, b]$, it follows that L is a countable set and R is a countable set. Let $w_1, w_2, w_3, \cdots$, denote a sequence of step-functions such that

(i) if x is in L and w is continuous from the left at x, then each of $w_1, w_2, w_3, \cdots$ is continuous from the left at x,

(ii) if x is in R and w is continuous from the right at x, then each of $w_1, w_2, w_3, \cdots$ is continuous from the right at x,

(iii) there is a positive number V such that if n is a positive integer, then $V_a^b(w_n) < V$, and

(iv) $w_1, w_2, w_3, \cdots$, converges to w uniformly in $[a, b]$.

Now for each positive integer n and each real number x in $[a, b]$, let $f_n(x)$ denote the number $\int_a^x v(t) dw_n(t)$ and let $f(x)$ denote the number $\int_a^x v(t) dw(t)$. Since v is in A, it follows from Lemma 4.1b that v is bounded in $[a, b]$. Let M denote the least upper bound of v in $[a, b]$. From the argument used in Case 1, it follows that $V_a^b(f_n) \leqq M V_a^b(w_n)$

$<MV$; hence by Lemma 4.2a, $f_1, f_2, f_3, \cdots$ converges to f uniformly in $[a, b]$. Again by Lemma 4.2a, it follows that $\int_a^b u(x)df_n(x) \to \int_a^b u(x) df(x)$ and $\int_a^b u(x)v(x)dw_n(x) \to \int_a^b u(x)v(x)dw(x)$ as $n \to \infty$. But if n is a positive integer, then $\int_a^b u(x)df_n(x) = \int_a^b u(x)v(x)dw_n(x)$; hence $\int_a^b u(x) df(x) = \int_a^b u(x)v(x)dw(x)$. This completes the proof.

REMARK 4.3. The above theorem, with integration by parts, can be used to evaluate integrals. For example, it is easy to show that if c is a real number and n is a positive integer, then $\int_0^c x^{n-1}dx = c^n/n$.

Remark added in proof. My attention has been called to the fact that the results in this paper overlap the results in a doctoral dissertation (University of Michigan, 1934) by H. S. Kaltenborn, *On Stieltjes mean integrals*, concerning the four kinds of integrals defined by H. L. Smith, *On the existence of the Stieltjes integral* (Trans. Amer. Math. Soc. vol. 27 (1925) pp. 491–515).

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