SOME NON-ABELIAN EXTENSIONS OF COMPLETELY DIVISIBLE GROUPS[1]

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1. **Introduction.** Baer [1; 2] has showed that those abelian groups G which are direct summands of every including abelian group are precisely those abelian groups G for which $nG=G$ for every positive integer n. The latter class of groups consists of the so-called complete or infinitely divisible abelian groups. Examination of the proof of the equivalence of these two classes discloses essential difficulties in the way of extension to the non-abelian case. Once "complete" is suitably defined for these latter groups, we can prove the following: Let H be a complete group interpolated into the ascending central series of a group G. Let K be a subgroup, maximal with respect to the property of meeting H on that portion of the ascending series of G below H. Then if $N(K)$ is the normalizer of K in G, $N(K)=H+K$. This result seems to be the natural extension of half of the Baer theorem to the non-abelian case.

Let us write all groups additively, whether they be abelian or not. For a subgroup H of a group G we let $N(H; G)$ be the normalizer of H in G. C_v is to be the cyclic subgroup of G generated by the element $v \in G$. $C(v; G)$ is to be the centralizer of v in G. For a subgroup H, $C(H; G)$ is to denote the centralizer of H in G. (See [4] for definitions.) Let 0 be the unity of G, and let (0) be the one element subgroup of G. In what follows, m, n, and r will always denote nonzero integers. $D(H; G)$, for a subset H of G, is to be the set of all $x \in G$ for which there exists $m=m(x)$ with $mx \in H$. $D(H; G)$, the *division-hull* of H in G, need not be a subgroup of G in the non-abelian case even if H is a subgroup.

We shall say that a group G is *complete* if, to each ordered pair (g, n), where $g \in G$, there exists a finite set of elements $g_i(g, n)=g_i$ $(i=1, 2, \cdots, m(g, n)=m)$ with $ng_1+ng_2+\cdots+ng_m=g$. If G is both abelian and complete we can always choose $m=1$.

For a group G, define in the customary fashion [4] the *ascending central series* $\{Z_i(G)\}$ $(i=0, 1, 2, \cdots)$, where $Z_0(G)=(0)$, $Z_1(G)$ is the center of G, and $Z_{i+1}(G)/Z_i(G)$ is the center of $G/Z_i(G)$. A subgroup H of G is said to be *interpolated in the ascending central series at* d if $Z_d(G) \subset H \subset Z_{d+1}(G)$ (where $\subset$ does not preclude equality), and d is the least integer for which this is true.

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If H and K are subsets of a group G, let $H+K$ be the set of all $h+k$, where $h\in H$ and $k\in K$. In general, $H+K$ need not be a subgroup, though it surely is a subgroup if both H and K are and at least one of these is normal. We note here that $\oplus$ denotes direct summation of subgroups.

2. The normalizer decomposition.

LEMMA 1. *Let H and K be a pair of subgroups of G where K is maximal with respect to the property of being disjoint from H. Then $N(K; G) \subset D(H+K; G)$.*

PROOF. We follow [1] and [3]. If $x\in N(K; G)$, $x\notin H+K$, then $x\notin K$. Form the subgroup $K'=\{K, x\}$ which has the generators x and all the elements of K. K' includes K properly. By the maximal character of K, one can find a nonzero element $h\in K'\cap H$. Since $x\in N(K; G)$, it is possible to find an integer t and an element $k\in K$ with $h=tx+k$. If $t=0$, then $h=k$; and $H\cap K=(0)$ then implies $h=0$, a contradiction. Hence $tx\in H+K$ with nonzero t, and $x\in D(H+K; G)$. Since also $(H+K)\cap N(K; G)\subset D(H+K; G)$, it follows that $N(K; G) \subset D(H+K; G)$.

THEOREM 1. *Let H be a complete group interpolated at d into the ascending central series of a group G. Let K be a subgroup of G, maximal with respect to the property of having precisely $Z_d(G)$ as its intersection with H. Then $N(K; G)=H+K$ and $N(K; G)/Z_d(G)\cong H/Z_d(G) \oplus K/Z_d(G)$.*

PROOF. We follow [3]. Suppose that $d=0$. Then H is included in the center of G, and $H\cap K=(0)$. For a given $x\in N(K; G)$, $x\notin H+K$, let r be the least positive integer (provided by Lemma 1) for which $rx\in H+K$. Let p be a prime divisor of r, and let $y=(r/p)x$. $py=rx =h+k$ for suitable $h\in H$ and $k\in K$. Since H is complete and abelian, there exists $h_1\in H$ with $ph_1=h$; and $-ph_1+py=k$. Since also $h_1\in Z_1(G)$, $-ph_1+py=p(-h_1+y)=k$. Let $z=-h_1+y$. Since $H \subset Z_1(G)$ and since $y=(r/p)x\in N(K; G)$, it follows that $z\in N(K; G)$. If $z\in H+K$, then $y=(r/p)x\in H+K$, contradicting the minimum character of r. Form the subgroup $K''=\{K, z\}$. K'' includes K properly, and by the maximal character of the latter subgroup there exists a nonzero $h'\in K''\cap H$. Since $z\in N(K; G)$ we can find an integer t and an element $k'\in K$ with $h'=tz+k'$. If $p|t$, $pz\in K$ implies $h'\in K$, contradicting $H\cap K=(0)$. Then there exist integers a and b for which $at+bp=1$, so that $z=atz+bpz$. But $tz\in H+K$ implies $atz\in H+K$ since $H\subset Z_1(G)$; and $pz\in K$. Thus $z\in H+K$, a contradiction. We have established that $r=1$, $x\in H+K$, and that $N(K;G)=H+K$ if $d=0$.

If $d\neq 0$, reduce the group modulo $Z_d(G)$. Let the images of G, H, and K be, respectively, G', H', and K'. It can be readily checked that K' is maximal in G' with respect to the property of being disjoint from H', that H' is in the center of G', and that H' is a complete abelian group. Using the case $d=0$ above, we have $N(K';G')=H'+K'$, and a trivial argument now shows that $N(K;G)=H+K$.

Subgroups which are interpolated into the ascending central series are normal subgroups, so that H is normal in G, and $H/Z_d(G)$ is normal in $G/Z_d(G)$. Hence $H/Z_d(G)$ is normal in $N(K;G)/Z_d(G)$. Moreover K is normal in $N(K;G)$ so $K/Z_d(G)$ is normal in $N(K;G)/Z_d(G)$. Since $H/Z_d(G)\cap K/Z_d(G)=(0)$, and since $N(K;G)/Z_d(G)=H/Z_d(G)+K/Z_d(G)$, we have proved that the sign $+$ in the last statement can be replaced by $\oplus$.

An immediate result is

Corollary 1. *Let a complete group H have an extension to a nilpotent group G of class $d+1$ in such a way that $Z_d(G)\subset H$. If K is any subgroup of G which is maximal with respect to the property of having precisely $Z_d(G)$ as its intersection with H, then K is normal in G, and $G=H+K$.*

Corlllary 2. *If H and K are as in the theorem and if $d=0$, then $N(K;G)\cong H\oplus K$.*

Corollary 3. *If H and K are as in the theorem and if $d=0$, then $N(K;G)/C(K;G)\cong J(K)$, the group of inner automorphisms of K.*

Proof. Since $C(K;G)\subset N(K;G)$ and since $N(K;G)=H+K$, every element of $C(K;G)$ has the form $h+k$, where $h\in H$ and $k\in K$. $h+k+k'=k'+h+k$ for every $k'\in K$. Since $H\subset Z_1(G)$, $k+k'=k'+k$ for every $k'\in K$, and $k\in Z_1(K)$. We can thus establish that $C(K;G)=H+Z_1(K)$. For $k\in K$, let γ_k be the inner automorphism $\gamma_k(x)=k+x-k$ for every $x\in K$. Define a map θ on $H+K$ into $J(K)$ as follows: $\theta(h+k)=\gamma_k$. Then it is easy to verify that θ is a homomorphism on $H+K$ onto $J(K)$ with kernel $H+Z_1(K)$.

One could ask whether there is anything to be said if H is a subgroup not necessarily complete or interpolated into the ascending central series. Let (A) be the property of a proper subgroup H of a group G that $mu+v$ (or, alternately, $v+mu)\in H$ where $u, v\in G$ and $C_v\cap H=(0)$ implies the existence of $g\in H\cap C(u;G)$ with $mg=mu+v$ (alternately, $v+mu$). Suppose that G is aperiodic and that H is a proper subgroup of G with property (A). Then it is easy to prove that

H is a normal subgroup of G and that $D(H;\ G)=H\subset\bigcap C(u;\ G)$, where the cross-cut is taken over all elements $u\in G$ such that $u\notin H$. H is likewise *strongly complete* in the sense that the equation $nx=h\in H$ always has a solution in H. Even if G has nontrivial periodic elements, a strongly complete subgroup H which is included in the center of G has property (A). The proofs of Lemma 1 and of Theorem 1 can be rewritten to give the somewhat weaker result:

THEOREM 2. *Let H be a subgroup with property* (A) *in a group G. Suppose that there exists a normal subgroup K of G which, as a subgroup, is maximal among the set of all subgroups* (*normal or not*) *which are disjoint from H. Then $G=H+K$; and if G is aperiodic, $G=H\oplus K$.*

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