DINI DERIVATIVES OF CONTINUOUS FUNCTIONS

ANTHONY P. MORSE

Let us agree that our space R is the set of real finite numbers.

Although it is clearly possible for the upper right-hand derivative D^+f of a function f continuous on R to R to assume only the values ± 1, nevertheless something of interest can be said about intermediary values assumed by the Dini derivatives. For example an almost immediate consequence of Theorem 5 below is Theorem 1.

1. Theorem. *If f is continuous on R to R, $-\infty < \lambda < \infty$, the set*

$$Ex[D^+f(x) \geqq \lambda]$$

is dense, the set

$$Ex[D^+f(x) < \lambda]$$

is nonvacuous, then the set

$$Ex[D^+f(x) = \lambda]$$

has the power of the continuum.

Our reasoning is based on a category argument.

Let us agree now that $f^*(A)$ is the image under the function f of the set A.

2. Lemma. *If C is a compact nondense set and f is such a continuous function on R to R that*

$$f(x) > f(y)$$

whenever x and y are such members of C that $x < y$, then the set $f^*(C)$ is compact and nondense.

Proof. Let g be such a strictly decreasing continuous function on R onto R that

$$g(x) = f(x) \qquad \text{whenever } x \in C.$$

Since g is a continuous one-one mapping of R onto itself, it follows that $g^*(C)$ is compact and nondense. In view of the fact that

$$f^*(C) = g^*(C)$$

the proof is complete.

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3. THEOREM. *If f is continuous on R to R, the set*

$$Ex[D^+f(x) \geqq 0]$$

is dense,

$$N = Ex[D^+f(x) < 0],$$

then the set

$$N \cup f^*(N)$$

is of the first category.

PROOF. Let ω be the set of non-negative integers. For $n \in \omega$ let

$$C_n = Ex[f(x+h) - f(x) \leqq -2^{-n} \cdot h \text{ whenever } 0 \leqq h \leqq 2^{-n}].$$

Since

$$N \cup f^*(N) = \bigcup_{n \in \omega} [C_n \cup f^*(C_n)]$$

the desired conclusion is a consequence of the statement.

STATEMENT. *If $n \in \omega$, then the set*

$$C_n \cup f^*(C_n)$$

is of the first category.

PROOF. Let J be such a sequence of closed intervals that:

$$\operatorname{diam} J_k = 2^{-n} \qquad \text{whenever } k \in \omega;$$

$$R = \bigcup_{k \in \omega} J_k.$$

After noticing that C_n is closed we check, for $k \in \omega$, that:

$$C_n \cap J_k$$

is compact and nondense; moreover

$$f(x) > f(y)$$

whenever x and y are such members of $C_n \cap J_k$ that $x < y$; consequently because of Lemma 2 the set

$$f^*(C_n \cap J_k)$$

is compact and nondense; and thus

$$(C_n \cap J_k) \cup f^*(C_n \cap J_k)$$

is compact and nondense.

Since

$$C_n \cup f^*(C_n) = \bigcup_{k\in\omega} \{(C_n \cap J_k) \cup f^*(C_n \cap J_k)\}$$

we now see that

$$C_n \cup f^*(C_n)$$

is of the first category.

4. THEOREM. *If f is continuous on R to R, the set*

$$Ex[D^+f(x) \geqq 0]$$

is dense,

$$-\infty < a < b < \infty, \qquad f(b) < f(a),$$
$$J = Ex[a < x < b], \qquad K = Ey[f(b) < y < f(a)],$$
$$Z = J \cap Ex[D^+f(x) = 0],$$

then the set

$$K \sim f^*(Z)$$

is of the first category.

PROOF. For $y \in K$ let

$$M(y) = \text{Sup}\,\{J \cap Ex[f(x) = y]\},$$

and let

$$N = Ex[D^+f(x) < 0],$$
$$Z' = Ex[x = M(y) \text{ for some } y \in K \sim f^*(N)].$$

Clearly

$$K \sim f^*(N) = f^*(Z') \tag{1}$$

and hence

$$D^+f(x) \geqq 0 \qquad \text{whenever } x \in Z'. \tag{2}$$

On the other hand if $x \in Z'$ then $x \in J$ and

$$f(t) < f(x) \qquad \text{whenever } t \text{ is such that } x < t < b.$$

Consequently

$$D^+f(x) \leqq 0 \qquad \text{whenever } x \in Z'$$

and thus, because of (2),

$$D^+f(x) = 0 \qquad \text{whenever } x \in Z'.$$

We conclude with the help of (1) and Theorem 3 that

$$Z' \subset Z, \qquad K \sim f^*(N) = f^*(Z') \subset f^*(Z),$$
$$K \sim f^*(Z) \subset K \cap f^*(N) \subset f^*(N),$$
$$K \sim f^*(Z) \text{ is of the first category.}$$

The proof is complete.

Since $K \cap B$ has the power of the continuum whenever K is such a nonvacuous open set that $K \sim B$ is of the first category, it is now easy to verify Theorem 5.

5. THEOREM. *If f is continuous on R to R, the set*

$$Ex[D^+f(x) \geqq 0]$$

is dense, then either f is strictly increasing or the set

$$Ex[D^+f(x) = 0]$$

has the power of the continuum.

As advertised, Theorem 1 follows easily by so defining g on R that

$$g(x) = f(x) - \lambda \cdot x \qquad \text{whenever } x \in R$$

and then applying Theorem 5 to g.

6. REMARKS.

.1 If f is a continuous function on R to R whose difference quotients are unbounded from above and below in each nonvacuous open set, then the set

$$J \cap Ex[D^+f(x) = \lambda]$$

has the power of the continuum whenever J is a nonvacuous open set and $\lambda \in R$.

.2 W. H. Young showed in Messinger of Mathematics vol. 38 (1908–09) that the Weierstrass nondifferentiable continuous function W has the property that the set

$$Ex[D^+W(x) < \infty]$$

is of the first category. He raised the question: Is it countable? In view of .1 it clearly isn't.

Actually

$$Ex[D^+f(x) < \infty]$$

is a first category set with the power of the continuum whenever f is a continuous nondifferentiable function.

.3 If we so define F on R that for $x \in R$

$$F(x) = D^+W(x) \quad \text{or} \quad 0$$

according as $|D^+W(x)| \leqq 1$ or $|D^+W(x)| > 1$, then F is a bounded Borelian function which is everywhere discontinuous and yet has the property that

$$J \cap Ex[F(x) = \lambda]$$

has the power of the continuum whenever J is a nonvacuous open set and $-1 \leqq \lambda \leqq 1$.

.4 If f is continuous on R to R and

$$Ex[D^+f(x) < 0]$$

has power less than the continuum, then f is a nondecreasing function.

.5 If f is a continuous nondecreasing function on R to R,

$$\begin{aligned} f'(x) &= 0 && \text{for almost all } x, \\ f(0) &< f(1), \\ 0 \leqq \lambda &\leqq \infty, \end{aligned}$$

then the set

$$Ex[D_+f(x) = \lambda]$$

has the power of the continuum.

.6 However, the Cantor function ψ has the property that the set

$$Ex[D^+\psi(x) = \lambda]$$

is void whenever $0 < \lambda < \infty$.

.7 Because of .6 it may be seen that Theorem 1 becomes false if "D^+" is replaced by "D_+".

UNIVERSITY OF CALIFORNIA, BERKELEY