ON A TAUBERIAN THEOREM BY O. SZÁSZ

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1. Renyi [1][1] has proved that if the real series $\sum_0^\infty a_n$ is summable Abel to s, that is

$$\lim_{x\uparrow 1} \sum_0^\infty a_n x^n = \lim_{x\uparrow 1} (1-x)\cdot \sum_0^\infty s_n x^n = s \quad (s_n = a_0 + \cdots + a_n) \tag{1.1}$$

and

$$\lim_{n\to\infty} \frac{|a_1| + 2|a_2| + \cdots + n\cdot|a_n|}{n+1} = l < +\infty,$$

then

$$\sum_0^\infty a_n = s. \tag{1.2}$$

Later O. Szász [3] has given the following generalization of Renyi's theorem: From (1.1),

$$\sum_{m=0}^n m\cdot(|a_m| - a_m) = O(n), \tag{1.3}$$

and

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{\nu=m+1}^n \nu\cdot(|a_\nu| - a_\nu) = 0, \quad \frac{n}{m} \to 1,\ n > m, \tag{1.4}$$

or (1.1) and

$$\lim_{m\to\infty} \left\{\frac{1}{n}\cdot \sum_{\nu=0}^n \nu\cdot(|a_\nu| - a_\nu) - \frac{1}{m}\cdot \sum_{\nu=0}^m \nu\cdot(|a_\nu| - a_\nu)\right\} = 0, \quad \frac{n}{m} \to 1,\ n > m, \tag{1.4'}$$

follow (1.2).

We shall see later that (1.3) is not necessary for the validity of the last theorem, and that (1.4), (1.4′) may be replaced by more general conditions. We shall obtain a similar result for Borel's method of summability.

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[1] Numbers in brackets refer to bibliography at the end of this note.

Our main result will be

THEOREM 1. *Necessary and sufficient conditions for the convergence of the real series* $\sum_0^\infty a_n$ *are* (1.1) *and one of the conditions*

$$\liminf_{m\to\infty} \frac{1}{m+1}\cdot \sum_{\nu=m+1}^{n} \nu a_\nu \geqq 0, \qquad \frac{n}{m}\to 1,\ n>m, \tag{1.5}$$

$$\liminf_{m\to\infty} \left\{\frac{1}{n}\cdot \sum_{\nu=0}^{n} \nu a_\nu - \frac{1}{m}\cdot \sum_{\nu=0}^{m} \nu a_\nu\right\} \geqq 0, \qquad \frac{n}{m}\to 1,\ n>m. \tag{1.5'}$$

It is easy to see that (1.5) *includes* (1.4) *and that* (1.5′) *includes* (1.4′).

PROOF OF THEOREM 1. First we shall prove that (1.1), (1.5), and (1.5′) are necessary. From (1.2) follows (1.1) and

$$t_n = \frac{s_0+s_1+\cdots+s_n}{n+1}\to s, \qquad n\to\infty.$$

Now

$$t_n - t_{n-1} = \frac{1}{n}\cdot\frac{1}{n+1}\cdot \sum_{m=0}^{n} m a_m, \qquad n\geqq 1, \tag{1.6}$$

and

$$\frac{1}{n}\cdot \sum_{\nu=0}^{n} \nu a_\nu = (n+1)\cdot(t_n - t_{n-1}) = s_n - t_n,$$

so that

$$\lim_{n\to\infty} \frac{1}{n}\cdot \sum_{0}^{n} \nu a_\nu = 0.$$

From this (1.5) and (1.5′) follow immediately.

We shall prove now that from (1.1) and (1.5) or (1.1) and (1.5′) follows (1.2).

It was proved by O. Szász [4] that from (1.1) follows

$$\lim_{x\uparrow 1} (1-x)\cdot \sum_{n=0}^{\infty} t_n\cdot x^n = s. \tag{1.7}$$

By subtracting (1.7) from (1.1) we get

$$\lim_{x\uparrow 1} (1-x)\cdot \sum_{1}^{\infty} \frac{a_1+2a_2+\cdots+na_n}{n+1}\cdot x^n = 0. \tag{1.8}$$

R. Schmidt has proved that if a real sequence $\{b_n\}$ is summable

Abel and $\liminf_{m\to\infty} (b_n - b_m) \geqq 0$, $n/m \to 1$, $n>m$, then $\{b_n\}$ is convergent. From (1.7), Schmidt's theorem applied to the sequence $\{(1/n)\cdot \sum_{\nu=0}^{n} \nu a_\nu\}$, and (1.5′) it follows that

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n+1} = 0,$$

and by Tauber's classical theorem, (1.2) follows from (1.1) and (1.5′). We have

$$s_n - s_m = \sum_{k=m+1}^{n} a_k = \sum_{k=m+1}^{n} \frac{ka_k}{k} \cdot$$

Abel's inequality yields

$$s_n - s_m \geqq \frac{1}{m+1} \cdot \min_{m<k\leqq n} \sum_{\nu=m+1}^{k} \nu a_\nu$$

and (1.5) leads to

$$\liminf_{m\to\infty} (s_n - s_m) \geqq 0, \qquad \frac{n}{m} \to 1, n > m,$$

thus we see, by Schmidt's theorem, that (1.2) follows from (1.1) and (1.5). Q.E.D.

2. A real series $\sum_0^\infty a_n$ is summable Borel to s if

$$\lim_{x\uparrow\infty} e^{-x}\cdot \sum_0^\infty \frac{s_n}{n!}\cdot x^n = s, \qquad s_n = a_0 + a_1 + \cdots + a_n. \tag{2.1}$$

It was proved by O. Szász [5] that from (2.1) follows

$$\lim_{x\uparrow\infty} e^{-x}\cdot \sum_0^\infty \frac{t_n}{n!}\cdot x^n = s. \tag{2.2}$$

By substracting (2.2) from (2.1) we obtain

$$\lim_{x\uparrow\infty} e^{-x}\cdot \sum_1^\infty \frac{a_1 + 2a_2 + \cdots + na_n}{n+1}\cdot \frac{x^n}{n!} = 0. \tag{2.3}$$

Hardy and Littlewood have proved that if a series $\sum_0^\infty c_n$ is summable Borel and $c_n = o(1/n^{1/2})$, then $\sum_0^\infty c_n$ is convergent. By applying this theorem to the series $t_0 + \sum_{n=1}^\infty (t_n - t_{n-1})$ and using (2.2) we get

THEOREM 2. *A necessary and sufficient condition for the convergence of* $\sum_0^\infty a_n$ *is that* $\sum_0^\infty a_n$ *should be summable Borel and that*

$$\lim_{n\to\infty} (a_1+2a_2+ \cdots +na_n)/(n+1)=0.$$

R. Schmidt [2] has proved that if the real series $\sum_0^\infty d_n$ is summable Borel and $\liminf_{m\to\infty} (d_{m+1}+ \cdots +d_n)\geqq 0$, $(n-m)/m^{1/2}\to 0$, $n>m$, then $\sum_0^\infty d_n$ is convergent. By using this theorem, (2.3), and Theorem 2 we obtain

THEOREM 3. *Necessary and sufficient conditions for the convergence of the real series* $\sum_0^\infty a_n$ *are* (2.1) *and one of the conditions*

$$\liminf_{m\to\infty} \frac{1}{m+1}\cdot \sum_{\nu=m+1}^{n} \nu a_\nu \geqq 0, \qquad \frac{n-m}{m^{1/2}}\to 0,\ n > m. \tag{2.4}$$

$$\liminf_{m\to\infty} \left\{\frac{1}{n}\sum_{\nu=0}^{n} \nu a_\nu - \frac{1}{m}\sum_{\nu=0}^{m} \nu a_\nu\right\} \geqq 0, \qquad \frac{n-m}{m^{1/2}}\to 0,\ n > m. \tag{2.4'}$$

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