ON THE CONVERGENCE OF A SOLUTION OF A DIFFERENCE EQUATION TO A SOLUTION OF THE EQUATION OF DIFFUSION[1]

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1. **Introduction.** Let $f(x)$ be a Lebesgue integrable function satisfying the l_c condition (see, e.g., Hardy [**3**, p. 359])

$$\int_0^t [f(x+u) + f(x-u) - 2c(x)]du = o(t)$$

for each x in $0 \leqq x \leqq 1$. If $a_n = 2\int_0^1 f(x) \sin n\pi x dx$, $n = 1, 2, \cdots$, then

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \, e^{-n^2\pi^2 t} \tag{1}$$

is the Fourier series solution in R: $0 < x < 1$, $t > 0$, of the partial differential equation of diffusion

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

Furthermore, (1) satisfies the boundary conditions $u(+0, t) = u(1-0, t) = 0$, $t > 0$ and as a consequence of §1 and §3 of Appendix II of [**3**] it also satisfies the initial condition $u(x, +0) = f(x)$ at every point of continuity of $f(x)$ in $0 < x < 1$ as well as the condition $u(x, +0) = (1/2)[f(x+0) + f(x-0)]$ at every point x where $f(x)$ possesses these one-sided limits. Moreover, from a slight modification of Hardy and Rogosinski [**4**, p. 66] on Abel summability of Fourier series and from Theorems 270 (due to M. L. Cartwright [**1**]) and 273 in Appendix V of [**3**] it follows that as $t \to 0+$,

$$\lim u(x, t) = f(x)$$

uniformly in any closed interval of continuity of $f(x)$. The uniformity of the limit goes back to the original theorems of Fejér [**2**].

Now for the remainder of this note let $f(x)$ be continuous in $0 \leqq x \leqq 1$ except for at most N points at which it may have a finite jump. (Then all the preceding remarks are still applicable.) Let M be a positive integer variable, $\Delta x = M^{-1}$, and $\Delta t = r(\Delta x)^2$. Then if

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$$b_n = b_n(M) = (2/M) \sum_{j=1}^{M-1} f(j/M) \sin (n\pi j/M), \quad n = 1, 2, \cdots, M - 1,$$

an analogue of the Fourier series (1)

$$U_M(x, t) = \sum_{n=1}^{M-1} b_n \sin n\pi x [1 - 4r \sin^2 (n\pi/2M)]^{tM^2/r} \tag{3}$$

is an exact solution (see, e.g., [7]) of a partial difference equation analogue of (2)

$$\begin{aligned} U_M(x, t + \Delta t) &- U_M(x, t) \\ &= r[U_M(x + \Delta x, t) + U_M(x - \Delta x, t) - 2U_M(x, t)] \end{aligned} \tag{4}$$

for (x, t), $(x+\Delta x, t)$, $(x-\Delta x, t)$, and $(x, t+\Delta t)$ in R. It also satisfies exactly the boundary conditions $U_M(0, t) = U_M(1, t) = 0$, $t \geqq 0$ and the initial condition $U_M(x, 0) = f(x)$ for those x in $0 < x < 1$ for which Mx is an integer.

When it converges, the series

$$V_M(x, t) = \sum_{n=1}^{\infty} c_n \sin n\pi x [1 - 4r \sin^2 (n\pi/2M)]^{tM^2/r}, \tag{5}$$

where $c_n = c_n(M)$, $n = 1, 2, \cdots$, also satisfies (4) but not, in general, the corresponding boundary conditions except when $c_n = b_n$ for $n \leqq M-1$ and $c_n = 0$ for $n \geqq M$.

Recent interest in the problem of convergence as $M \to \infty$ has resulted in a proof by Leutert [6] of the convergence of (5) to (1) for the case where $0 < r \leqq 1/4$ and $|c_n - a_n| \to 0$ uniformly for $n < M$ and for $f(x)$ sectionally continuous with one-sided derivatives everywhere. Hildebrand [5] proved convergence of (3) to (1) for $0 < r \leqq 1/2$ when $f(x)$ has bounded variation and is continuous except for a finite number of finite jumps. He had more severe restrictions on $f(x)$ when $r = 1/2$. With our considerably more general $f(x)$ we shall prove convergence of (3) to (1) for $0 < r \leqq 1/2$ and convergence of (5) to (1) for all $r > 0$.

2. **Two lemmas.** Let β be an arbitrary, fixed number in $0 < \beta < 1$ and define

$$k = k(M, r, \beta) = \begin{cases} \langle M/3 \rangle, & 0 < r \leqq 1/2, \\ \langle (2M/\pi) \sin^{-1} (\beta/4r)^{1/2} \rangle, & r > 1/2, \end{cases} \tag{6}$$

where $\langle x \rangle$ represents the greatest integer not exceeding x.

Lemma 1. *Uniformly for all positive integers M and all $t \geqq t_0 > 0$,* *we have, for* $0<r\leqq 1/2$,

$$\sum_{n=1}^{M-1} \left| 1 - 4r \sin^2 (n\pi/2M) \right|^{tM^2/r} = O(1) \tag{7}$$

and, for $r>0$,

$$\sum_{n=1}^{k} \left| 1 - 4r \sin^2 (n\pi/2M) \right|^{tM^2r} = O(1) \tag{8}$$

where k *is defined by* (6).

PROOF. Let $0<r\leqq 1/2$. Then, for $n\leqq \langle M/2\rangle$,

$$\begin{aligned} - 1 + 4r \sin^2 (n\pi/2M) & \\ \leqq - 1 + 4r \sin^2 \frac{M-n}{2M}\pi &= - 1 + 4r \cos^2 (n\pi/2M) \\ &= 4r - 1 - 4r \sin^2 (n\pi/2M) \leqq 1 - 4r \sin^2 (n\pi/2M). \end{aligned} \tag{9}$$

Using (9) and the inequalities $\log (1-z)\leqq -z$ for $0\leqq z<1$ and $2z/\pi\leqq \sin z$ for $0\leqq z\leqq \pi/2$, we have

$$\begin{aligned} \sum_{n=1}^{M-1} \left| 1 - 4r \sin^2 (n\pi/2M) \right|^{tM^2/r} & \\ &\leqq 2 \sum_{n=1}^{\langle M/2\rangle} [1 - 4r \sin^2 (n\pi/2M)]^{tM^2/r} \\ &\leqq 2 \sum_{n=1}^{\langle M/2\rangle} e^{-4M^2 t \sin^2 (n\pi/2M)} < 2 \sum_{n=1}^{\langle M/2\rangle} e^{-4n^2 t} = O(1). \end{aligned}$$

For the larger range of r, $r>0$, (8) is proved using the same inequalities in the same manner.

LEMMA 2. *For* $n\leqq k$, $r>0$, *and* $t\geqq t_0>0$,

$$\delta \equiv \left| [1-4r \sin^2 (n\pi/2M)]^{tM^2/r} - e^{-n^2\pi^2 t} \right| \leqq Ct(n^4\pi^4/4M^2)e^{-n^2\alpha^2\pi^2 t} \tag{10}$$

where $C=\max \{1/3,\ 2r/(1-\beta)\}$ *and*

$$\alpha^2=1-\max \{\pi^2/108,\ (1/3)\ [\sin^{-1} \beta/4r]^2\}.$$

PROOF. By the Mean Value Theorem we have, for $n\leqq k$,

$$\delta = n^2\pi^2t \left| (M^2/n^2\pi^2r) \log [1 - 4r \sin^2 (n\pi/2M)] + 1 \right| e^{-\xi\pi^2 t}$$

for some ξ lying between n^2 and $-(M^2/\pi^2r) \log [1-4r \sin^2(n\pi/2M)]$. From the elementary inequalities, $-z-z^2/2(1-z)\leqq \log (1-z)\leqq -z$ for $0\leqq z<1$, $\sin z\leqq z$ for $z\geqq 0$, and $\sin^2 z\geqq z^2-z^4/3$ for $|z| <\pi/2$,

$$
\begin{aligned}
\frac{-2r}{1-\beta}\left(\frac{n\pi}{2M}\right)^2 &\leqq \frac{-2r}{1-4r\sin^2(n\pi/2M)}\left(\frac{n\pi}{2M}\right)^2 \\
&= -\frac{1}{r}\left(\frac{M}{n\pi}\right)^2\left[4r\left(\frac{n\pi}{2M}\right)^2+\frac{8r^2(n\pi/2M)^4}{1-4r\sin^2(n\pi/2M)}\right]+1 \\
&\leqq \left(\frac{M}{n\pi}\right)^2\frac{1}{r}\log\left[1-4r\sin^2(n\pi/2M)\right]+1 \qquad (11)\\
&\leqq -4(M/n\pi)^2\sin^2(n\pi/2M)+1 \\
&\leqq -(2M/n\pi)^2\left[(n\pi/2M)^2-(1/3)(n\pi/2M)^4\right]+1 \\
&= (1/3)(n\pi/2M)^2.
\end{aligned}
$$

The last inequality of (11) yields

$$\xi \geqq n^2\left[1-(1/3)(n\pi/2M)^2\right]. \qquad (12)$$

From (11) and (12) we obtain (10).

3. **Convergence theorems.**

THEOREM 1. *For any fixed* $t_0>0$ *and for* $0<r\leqq 1/2$,

$$\lim_{M\to\infty} U_M(x,t) = u(x,t)$$

uniformly for $0\leqq x\leqq 1$ *and* $t\geqq t_0$.

PROOF. For $k=1, 2, \cdots$, let

$$\sigma_k(x,t) = \sum_{n=1}^{k} A_{k,n}\sin n\pi x e^{-n^2\pi^2 t}$$

be the arithmetic mean of the first k partial sums of (1). Then

$$A_{k,n} = \begin{cases}(k-n+1)a_n/k, & 1\leqq n\leqq k,\\ 0, & n>k.\end{cases}$$

For k as defined in (6) we have

$$|U_M(x,t)-u(x,t)| \leqq |E_1(x,t)|+|E_2(x,t)|+|E_3(x,t)| \qquad (13)$$

where

$$
\begin{aligned}
E_1(x,t) &= \sum_{n=1}^{M-1}(b_n-A_{k,n})\sin n\pi x\left[1-4r\sin^2(n\pi/2M)\right]^{tM^2/r}, \\
E_2(x,t) &= \sum_{n=1}^{k}A_{k,n}\sin n\pi x\left\{\left[1-4r\sin^2(n\pi/2M)\right]^{tM^2/r} - e^{-n^2\pi^2 t}\right\}, \qquad (14)
\end{aligned}
$$

and

$$E_3(x, t) = \sum_{n=1}^{k} A_{k,n} \sin n\pi x e^{-n^2\pi^2 t} - u(x, t). \tag{15}$$

Let us cover the set of points of discontinuity of $f(x)$ by a set E which is the sum of a finite number of open intervals, the sum, η, of whose lengths is arbitrarily small. Let I be the interval $0 \leqq x \leqq 1$. Then, on $I-IE$, by the corollary to Fejér's principal theorem on summability $(C, 1)$ of Fourier series (see Fejér [2, p. 60]), $\sigma_k(x, 0)$ converges uniformly to $f(x)$. On IE, $|\sigma_k(x, 0) - f(x)| \leqq 2F$ where $F = \text{LUB } |f(x)|$ in $0 \leqq x \leqq 1$. Then, for all sufficiently large M we have

$$\begin{aligned} |b_n - A_{k,n}| &= \frac{2}{M}\left| \sum_{j=1}^{M-1} [f(j/M) - \sigma_k(j/M, 0)] \sin (n\pi j/M) \right| \\ &< (2/M)[M \cdot o(1) + 2F(\eta M + N)] = o(1) \end{aligned} \tag{16}$$

uniformly for all $n \leqq M$. Hence, from (16) and (7) of Lemma 1, we have that the first member of the right-hand side of (13) is $o(1)$ uniformly for $0 \leqq x \leqq 1$ and $t \geqq t_0$.

Using $|A_{k,n}| \leqq |a_n| \leqq 4F/\pi$ and Lemma 2, we have

$$|E_2(x, t)| \leqq (FCt\pi^3/M^2) \sum_{n=1}^{k} n^4 e^{-n^2\alpha^2\pi^2 t} = O(M^{-2})$$

uniformly in $0 \leqq x \leqq 1$ and $t \geqq t_0$.

Finally, since for sufficiently large k

$$\begin{aligned} |A_{k,n} - a_n| &\leqq 2\int_0^1 |\sigma_k(y, 0) - f(y)| \, |\sin n\pi y| \, dy \\ &\leqq (4/\pi)[o(1) + 2F\eta] = o(1), \end{aligned} \tag{17}$$

we have

$$|E_3(x, t)| \leqq o(1) \cdot \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} = o(1)$$

uniformly for $0 \leqq x \leqq 1$ and $t \geqq t_0$, completing the proof of the theorem.

THEOREM 2. *Let $c_n = c_n(M)$ in (5) be such that $c_n = 0$ for $n > k$ (where k is defined by (6)) and*

$$\lim_{M\to\infty} |c_n - a_n| = 0 \tag{18}$$

uniformly for $n \leqq k$. Then, for each $r > 0$,

$$\lim_{M\to\infty} V_M(x, t) = u(x, t)$$

uniformly in $0 \leqq x \leqq 1$ *and* $t \geqq t_0 > 0$.

PROOF: We have

$$| V_M(x, t) - u(x, t) | \leqq | E_1^*(x, t) | + | E_2(x, t) | + | E_3(x, t) |$$

where $E_2(x, t)$ and $E_3(x, t)$ are defined by (14) and (15) and

$$E_1^*(x, t) = \sum_{n=1}^{k} (c_n - A_{k,n}) \sin n\pi x[1 - 4r \sin^2 (n\pi/2M)]^{tM^2/r}.$$

Using (17) and (18) in the triangle inequality, we get $c_n - A_{k,n} = o(1)$. Using this and (8) of Lemma 1, we obtain $E_1^*(x, t) = o(1)$ uniformly for $0 \leqq x \leqq 1$ and $t \geqq t_0 > 0$. From the proof of Theorem 1, we have $E_2(x, t) + E_3(x, t) = o(1)$ uniformly for $0 \leqq x \leqq 1$ and $t \geqq t_0$, thus completing the proof of the theorem.

In the case $0 < r \leqq 1/2$, $U_M(x, 0)$ satisfies the initial conditions on a set asymptotically dense on $0 \leqq x \leqq 1$ as $M \to \infty$. On the other hand, (18) is not sufficient for $V_M(x, 0+)$ to satisfy the initial conditions or even be bounded. This is easily seen in the case where $c_n(M) = a_n + (1/M) \sin (n\pi/2)$ for $n \leqq M-1$ and $c_n(M) = 0$ for $n \geqq M$. However, $c_n = a_n + o(1/M)$ is clearly sufficient for convergence for $t = 0$.

Theorem 1 assures uniform convergence of $U_M(x, t)$ to $u(x, t)$ for $t \geqq t_0$ and $0 \leqq x \leqq 1$. However, $u(x, t)$ is real for every (x, t) in R and on its boundaries, while for $1/4 < r \leqq 1/2$ and those (x, t) such that tM^2/r is not an integer, $U_M(x, t)$ may be complex. Therefore, we shall state two more theorems covering a general class of real interpolations on $U_M(x, t)$ and $V_M(x, t)$, which include, e.g., bilinear interpolation. Let P_1, P_2, P_3, and P_4 be the points at the corners of an elemental rectangle of area $\Delta x \Delta t$. Then, if $\alpha_i(x, t)$, $i = 1, \cdots, 4$, are non-negative functions whose sum is unity and if $\overline{W}_M(x, t) = \sum_{i=1}^{4} \alpha_i(x, t) W_M(P_i)$, then we say $\overline{W}_M(x, t)$ is a four-point interpolation on $W_M(x, t)$ satisfying a "maximum-minimum principle." In our definition we also assume that, for (x, t) on the boundary of an elementary rectangle, $\overline{W}_M(x, t)$ is determined solely by interpolation on the two neighboring meshpoints determining the straight-line segment of the boundary containing (x, t).

THEOREM 3. *If* $\overline{U}_M(x, t)$ *is a four-point interpolation on* $U_M(x, t)$ *satisfying a "maximum-minimum principle," then*

$$\lim_{M\to\infty} \overline{U}_M(x, t) = u(x, t) \tag{19}$$

uniformly in $0 \leqq x \leqq 1$ *and* $t \geqq t_0 > 0$. *Furthermore, uniformly on* $I - IE$

$$\lim_{M \to \infty} \overline{U}_M(x, 0) = f(x).$$

PROOF. For any P_1, P_2, P_3, and P_4 as defined above we have

$$\left| \overline{U}_M(x, t) - u(x, t) \right| \leqq \sum_{i=1}^{4} \alpha_i \left| U_M(P_i) - u(P_i) \right| + \sum_{i=1}^{4} \alpha_i \left| u(P_i) - u(x, t) \right|.$$

Then, (19) follows immediately from Theorem 1 and the uniform continuity of $u(x, t)$ for $t \geqq t_0$. Similarly, if $(x_1, 0)$ and $(x_2, 0)$ are adjacent meshpoints such that $x_1 \leqq x \leqq x_2$ is in $I - IE$, then, since $\overline{U}_M(x_i, 0) = f(x_i)$ and $f(x)$ is uniformly continuous on $I - IE$, we have

$$\left| \overline{U}_M(x, 0) - f(x) \right| \leqq \sum_{i=1}^{2} \alpha_i \left| f(x_i) - f(x) \right| = o(1).$$

Similarly, one can prove the following theorem.

THEOREM 4. *If* $\overline{V}_M(x, t)$ *is a four point interpolation on* $V_M(x, t)$ *satisfying a "maximum-minimum" principle and if* (18) *holds, then*

$$\lim_{M \to \infty} \overline{V}_M(x, t) = u(x, t)$$

uniformly in $0 \leqq x \leqq 1$ *and* $t \geqq t_0 > 0$.

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