THE BOUNDEDNESS OF THE SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION

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1. **Introduction.** In this note we consider the differential equation

$$(r(x)y')' + q(x)y = f(x, y) \tag{1}$$

with a "small" nonlinear term $f(x, y)$. The exact conditions satisfied by r, q, and f will be given in §§2 and 3.

If r, q, and $(rq)'$ are real-valued continuous functions on the positive x-axis, in a recent paper [1] Walter Leighton proved that the following conditions $rq>0$ and $(rq)'\geqq 0$ for $x\geqq 0$ are sufficient for the boundedness of the solutions of the linear differential equation $(ry')'+qy=0$ on the positive x-axis. It is the purpose of this note to point out that, even if rq is not differentiable (in fact, even the continuity of rq is not required), under certain suitable conditions satisfied by r, q, and f as stated in Theorems 1 and 2, we still can test the boundedness of the solutions of (1) by a method which includes Leighton's method as a special case. In §2 we assume that r, q, and f are real-valued functions; in §3, complex-valued.

2. **Boundedness of the solutions.** Throughout this section we assume that $1/r(x)$, $q(x)$, and $f(x, y)$ (for each fixed y) are real-valued functions defined for all $x\geqq 0$ and belonging to $L(0, R)$ for every positive R. Furthermore we assume that for each $x\geqq 0$, $f(x, y)$ is a continuous function of y satisfying the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leqq g(x)|y_1 - y_2|, \tag{2}$$

where $g(x)$ and $f(x, 0)$ belong to $L(0, \infty)$.

By a solution of (1) we mean a function $y(x)$ which is absolutely continuous and $r(x)y'(x)$ is equal to an absolutely continuous function $z(x)$, say, almost everywhere such that, with $z(x)$ replacing $r(x)y'(x)$, (1) is satisfied almost everywhere on $0\leqq x<\infty$. For the existence and uniqueness of solutions, see [2, sections 68.3 and 68.5].

Denoting max $(u'(x), 0)$ by $(u(x))'_+$ and min $(u'(x), 0)$ by $(u(x))'_-$, the theorem we shall establish is

THEOREM 1. *If there exists a real-valued function $p=p(x)$ such that*

1. *p belongs to $L(0, R)$ for every positive R,*

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2. *rp is positive and absolutely continuous on* $0 \leqq x < \infty$,

3. *$p-q$ and $(rp)^{-1}(rp)'_-$ belong to* $L(0, \infty)$,

then every solution of (1) *is bounded on* $0 \leqq x < \infty$.

PROOF. First we prove that every solution of

$$(r(x)y')' + p(x)y = 0 \tag{3}$$

is bounded on $0 \leqq x < \infty$. This can be done by modifying Leighton's proof [1]. Multiplying (3) by ry' and integrating from 0 to x, we obtain

$$(ry')^2 + rpy^2 = c + \int_0^x (rp)'y^2dx, \tag{4}$$

where c is a positive constant. From (4), clearly,

$$rpy^2 \leqq c + \int_0^x (rp)'_+y^2dx, \qquad x \geqq 0. \tag{5}$$

Multiplying both sides of (5) by $(rp)^{-1}(rp)'_+$ and then dividing by $c+\int_0^x(rp)'_+y^2dx$, we have

$$\frac{(rp)'_+y^2}{c + \int_0^x (rp)'_+y^2dx} \leqq (rp)^{-1}(rp)'_+, \qquad x \geqq 0. \tag{6}$$

Integration of (6) from 0 to x gives

$$\log\left(c + \int_0^x (rp)'_+y^2dx\right) \leqq \log c + \int_0^x (rp)'_+(rp)^{-1}dx, \qquad x \geqq 0. \tag{7}$$

(5) and (7) then yield

$$rpy^2 \leqq c \exp\left(\int_0^x (rp)'_+(rp)^{-1}dx\right), \qquad x \geqq 0. \tag{8}$$

Since $(rp)'_+ = (rp)' - (rp)'_-$, a simple calculation shows that

$$r(0)p(0)y^2 \leqq c \exp\left(-\int_0^x (rp)'_-(rp)^{-1}dx\right), \qquad x \geqq 0. \tag{9}$$

In view of the convergence of the integral in (9), y is bounded on $0 \leqq x < \infty$.

To prove the boundedness of the solutions of (1), we consider the integral equation

$$(10) \quad y(x) = Az_1(x) + Bz_2(x) + \int_0^x h(t, y(t))[z_2(x)z_1(t) - z_1(x)z_2(t)]dt,$$

where

$$h(x, y) = (p(x) - q(x))y + f(x, y)$$

and $z_1(x)$ and $z_2(x)$ are two linearly independent solutions of (3) with

$$r(x)[z_1(x)z_2'(x) - z_1'(x)z_2(x)] = 1$$

almost everywhere and A and B are arbitrary constants. Clearly $h(x, y)$ satisfies

$$(11) \qquad |h(x, y_1) - h(x, y_2)| \leqq k(x)|y_1 - y_2|,$$

where $k(x) = |p(x) - q(x)| + g(x)$, and $k(x)$ and $h(x, 0) = f(x, 0)$ belong to $L(0, \infty)$. Obviously (11) implies that $|h(x, y)| \leqq k(x)|y| + |h(x, 0)|$, and so the integral in (10) exists for every continuous function $y(x)$. We now show that (10) has a solution by successive approximations. Define $y_0(x) = 0$ and

$$(12) \qquad \begin{aligned} y_n(x) &= Az_1(x) + Bz_2(x) \\ &+ \int_0^x h(t, y_{n-1}(t))[z_2(x)z_1(t) - z_1(x)z_2(t)]dt \end{aligned}$$

for $n = 1, 2, 3, \cdots$. If $|z_2(x)z_1(t) - z_1(x)z_2(t)| \leqq N$ for $x \geqq 0$ and $t \geqq 0$ and if M is an upper bound of $|y_1(x)|$ on $0 \leqq x < \infty$, using (12) it is easy to verify that

$$(13) \qquad |y_{n+1}(x) - y_n(x)| \leqq M\frac{[H(x)]^n}{n!}, \qquad H(x) = N\int_0^x k(t)dt,$$

for $n = 0, 1, 2, \cdots$. From this it follows that $y_n(x)$ converges to a limit $y(x)$ uniformly on $0 \leqq x \leqq x_0$ for every x_0, and $y(x)$ satisfies (10) and

$$(14) \qquad |y(x)| \leqq M \exp(H(x)), \qquad x \geqq 0.$$

The boundedness of $y(x)$ then follows from the convergence of $H(x)$ and (14). Substitution of (10) into (1) shows that $y(x)$ satisfies (1) and so is a solution of (1). From the uniqueness of the solutions of (1), it follows that for different constants A and B, $y(x)$ represents all the solutions of (1). This proves that every solution of (1) is bounded on $0 \leqq x < \infty$.

We may also observe that, in view of the boundedness of $y(x)$, (10) can be written in the form

$$y(x) = c_1 z_1(x) + c_2 z_2(x) \\ - \int_x^\infty h(t, y(t)) [z_2(x) z_1(t) - z_1(x) z_2(t)] dt, \tag{15}$$

the integral part approaching zero as limit as x approaches ∞. Hence to each solution $y(x)$ of (1), there corresponds a solution $c_1 z_1(x) + c_2 z_2(x)$ of (3) such that their difference tends to zero as x tends to ∞. In fact the correspondence is one-to-one. This can be established by proving also the existence and uniqueness of the solution of (15) for two arbitrary constants c_1 and c_2. Differentiation of (15) also yields

$$y'(x) = (c_1 + \epsilon_1(x)) z_1'(x) + (c_2 + \epsilon_2(x)) z_2'(x) \tag{16}$$

for almost all x on $0 \leqq x < \infty$, where $\epsilon_1(x)$ and $\epsilon_2(x)$ tend to zero as x tends to ∞.

3. **Extension.** In this section we assume that $r(x)$, $q(x)$, and $f(x, y)$ satisfy the same conditions as stated in the first paragraph of §2 except that they are complex-valued functions. Denoting by a^* the conjugate of a, Theorem 1 can be slightly extended to

THEOREM 2. *If there exists a complex-valued function $p = p(x)$ such that*

1. *p belongs to $L(0, R)$ for every positive R,*
2. *rp^* is real, positive and absolutely continuous on $0 \leqq x < \infty$,*
3. *$p - q$ and $(rp^*)^{-1}(rp^*)'_-$ belong to $L(0, \infty)$,*

then every solution of (1) *is bounded on* $0 \leqq x < \infty$.

PROOF. Multiplying (3) by $(ry')^*$ and its conjugate by ry', we have the sum

$$(ry')^*(ry')' + (ry')(ry')'^* + r^* p y y'^* + r p^* y^* y' = 0. \tag{17}$$

Integration of (17) from 0 to x gives

$$|ry'|^2 + rp^* |y|^2 = c + \int_0^x (rp^*)' |y|^2 dx, \tag{18}$$

where c is a positive constant. The rest of the proof is similar to that of Theorem 1. The remarks at the end of §2 also apply here.

REFERENCES

1. Walter Leighton, *Bounds for the solutions of a second-order linear differential equation*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 190–191.

2. Edward J. McShane, *Integration*, Princeton, 1947.

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