

SOME THEOREMS ON n -HOMOGENEOUS CONTINUA

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1. **Introduction and definitions.** Bing [2] has given an example of a homogeneous indecomposable continuum in the plane. His example is homeomorphic with an example given by Moise [16] to show the existence of a nondegenerate plane continuum which is not an arc and is homeomorphic with each of its nondegenerate subcontinua. Whyburn [19, Theorem 3] has shown that such a continuum does not separate the plane. Jones [11, Theorem 2] has shown that every homogeneous bounded plane continuum which does not separate the plane is indecomposable. The purpose of this paper is to give some conditions under which a homogeneous continuum is indecomposable and to show that every bounded plane continuum possessing a certain type of homogeneity is a simple closed curve.

If M is a continuum and there exist n continua whose sum is M such that no one of them is a subset of the sum of the others, then M is said to be the *finished sum* of these n continua.

If the continuum M is the finished sum of n continua and is not the finished sum of $n+1$ continua, then M is said to be *indecomposable¹ under index n* .

A point set M is said to be *n -homogeneous* if for any n points x_1, x_2, \dots, x_n of M and any n points y_1, y_2, \dots, y_n of M there is a homeomorphism T of M onto itself such that $T(x_1+x_2+\dots+x_n) = y_1+y_2+\dots+y_n$. If in addition to these requirements, $T(x_i) = y_i$ ($1 \leq i \leq n$), then M is said to be *strongly n -homogeneous*. A point set which is 1-homogeneous is said to be *homogeneous*.

A point set M is said to be *nearly n -homogeneous* if for any n points x_1, x_2, \dots, x_n of M and any n open subsets² D_1, D_2, \dots, D_n of M there exist n points y_1, y_2, \dots, y_n of D_1, D_2, \dots, D_n respectively and a homeomorphism T of M onto itself such that $T(x_1+x_2+\dots+x_n) = y_1+y_2+\dots+y_n$. A set which is nearly 1-homogeneous is said to be *nearly homogeneous*.

A point set M is said to be *nearly homogeneous over a subset H of M* if for any point x of H and any open subset D of M there is a homeomorphism of M onto itself that carries x into a point of D .

It is obvious that every n -homogeneous set is nearly n -homogeneous.

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¹ Some properties of such continua are given in [8] and [18].

² A subset of M open relative to M is called an *open subset of M* .

ous and that every nearly n -homogeneous set is nearly homogeneous.

2. Homogeneous continua in a metric space. Theorems 1–6 are proved on the basis of R. L. Moore's³ Axioms 0 and 1₃. Hence these theorems hold true in any metric space.

THEOREM 1. *Every n -homogeneous point set M is homogeneous.*

PROOF. Suppose M is not homogeneous. Then there are two points x_1 and y_1 of M such that there does not exist a homeomorphism of M onto itself that carries x_1 into y_1 . Let x_2, x_3, \dots, x_n be distinct points of M different from x_1 and y_1 . There is a homeomorphism T of M onto itself such that $T(x_1 + x_2 + \dots + x_n) = y_1 + x_2 + x_3 + \dots + x_n$. Since $T(x_1) \neq y_1$, then there is a positive integer i_1 ($1 < i_1 \leq n$) such that $T(x_{i_1}) = x_{i_1}$. If $T(x_{i_1}) \neq y_1$, then there is a positive integer i_2 ($1 < i_2 \leq n$) different from i_1 such that $T(x_{i_2}) = x_{i_2}$. This process can be continued to obtain a positive integer r such that $1 < i_r \leq n$ and $T(x_{i_r}) = y_1$. Since T^{r+1} is a homeomorphism of M onto itself carrying x_1 into y_1 , then M is homogeneous.

THEOREM 2. *If $n > 1$ and the compact continuum M is indecomposable under index n , then M is not nearly homogeneous.*

PROOF. Suppose, on the contrary, that M is nearly homogeneous. By a theorem proved by Swingle [18, Theorem 2], M is the finished sum of n indecomposable continua M_1, M_2, \dots, M_n such that M_1 and M_2 have a point x in common. Let D be an open subset of M which intersects M_1 and does not intersect $M_2 + M_3 + \dots + M_n$. There is a homeomorphism T of M onto itself that carries x into a point of D . Then M is the finished sum of the indecomposable continua $T(M_1), T(M_2), \dots, T(M_n)$. Since $T(x)$ lies in both $T(M_1)$ and $T(M_2)$ and does not lie in $M_2 + M_3 + \dots + M_n$, then one of the continua $T(M_1)$ and $T(M_2)$ is not one of the continua M_1, M_2, \dots, M_n . This is contrary to Theorem 2 of [8].

THEOREM 3. *If $n > 1$ and the compact continuum M is nearly homogeneous over a set H consisting of n points about which M is irreducible, then there is a positive integer r not greater than n such that M is indecomposable under index r .*

PROOF. An inductive argument will be used. It will first be shown that Theorem 3 holds true if $n = 2$. Let x and y be the two points of H . Suppose that M is the finished sum of three continua M_1, M_2 , and M_3 . Consider the case in which $x + y$ is a subset of $M_1 + M_2$. Then

³ Axiom 1₃ is the first three parts of Axiom 1 as stated in [17].

$M_1 + M_2$ is not connected and M_3 intersects each of the continua M_1 and M_2 . There is a homeomorphism T of M onto itself that carries x into a point of $M - (M_1 + M_2)$. Then one of the continua $M_1 + M_3$ and $M_2 + M_3$ contains both of the points $T(x)$ and $T(y)$. This is impossible since each of these continua is a proper subset of M and M is irreducible between the points $T(x)$ and $T(y)$. Thus the supposition that M is the finished sum of three continua has led to a contradiction. Hence Theorem 3 holds true if $n = 2$.

Now let k be a positive integer such that Theorem 3 holds true if $n = k$. For the remainder of this argument it will be assumed that H consists of $k + 1$ points. Suppose that there is a collection G consisting of $k + 2$ continua whose finished sum is M . Then there is a continuum X of G such that the set⁴ $M - (G - X)^*$ contains no point of H . Let z be a point of H . There is a homeomorphism T of M onto itself that carries z into a point of $M - (G - X)^*$. Since $T(H) - T(z)$ consists of k points, then some proper subcontinuum M' of M contains $T(H) - T(z)$ but not $T(z)$. Since $M' + (G - X)^*$ is a proper subset of M containing H , then $M' + (G - X)^*$ is not connected. Hence some continuum L of the collection $G - X$ does not intersect M' . Since $(G - L)^*$ is a proper subset of M containing $T(H)$, then $(G - L)^*$ is not connected. Hence $(G - L)^*$ is the sum of two mutually separated sets L_1 and L_2 such that L_1 contains the continuum M' . Since $L_1 + L$ is a proper subcontinuum of M , then L_2 contains $T(z)$. Consequently L_2 contains the continuum X . Since the closed set $(G - X)^*$ contains H , then $(G - X)^*$ is the sum of two mutually separated sets X_1 and X_2 such that X_1 contains the continuum $L_1 + L$. Hence $X_1 + X$ contains $T(H)$. This is impossible since $X_1 + X$ is a proper subcontinuum of M and M is irreducible about $T(H)$. Thus the supposition that M is the finished sum of $k + 2$ continua has led to a contradiction. This completes the inductive argument.

THEOREM 4. *If the nondegenerate compact continuum M is nearly homogeneous and is irreducible about some finite set, then M is indecomposable.*

This theorem follows from Theorems 2 and 3.

COROLLARY. *If the nondegenerate compact continuum M is nearly homogeneous and is homeomorphic with each of its nondegenerate subcontinua, then M is indecomposable.⁵*

⁴ The collection of all elements of G different from X is denoted by $G - X$, and the sum of the sets of $G - X$ is denoted by $(G - X)^*$.

⁵ If M is a homogeneous plane continuum this follows from [11, Theorem 2] and [19, Theorem 3].

THEOREM 5. *If $n > 1$, then in order that the compact continuum M should be indecomposable under index n it is necessary and sufficient that (1) M should contain a set H consisting of n points such that M is aposyndetic⁶ at each point x of H with respect to each point of $H - x$ and (2) M should not contain $n + 1$ such points.*

PROOF OF NECESSITY. By [18, Theorem 2], M is the finished sum of n indecomposable continua M_1, M_2, \dots, M_n . Let H be a set consisting of n points x_1, x_2, \dots, x_n such that for each i ($i \leq n$), x_i is a point of M_i and does not belong to M_j ($j \neq i$). Then H is aposyndetic at each point of H with respect to each other point of H . Now suppose that there exists a set K consisting of $n + 1$ points of M such that M is aposyndetic at each point of K with respect to each other point of K . Then there is a positive integer i such that M_i contains two points y_1 and y_2 of K . There is a continuum L and an open subset U of M such that $M - y_1 \supset L \supset U \supset y_2$. This is contrary to Lemma 1.2 of [7].

PROOF OF SUFFICIENCY. Since M does not contain $n + 1$ points such that M is aposyndetic at each of them with respect to each of the others, then M is not the finished sum of $n + 1$ continua. Hence there is a positive integer r not greater than n such that M is indecomposable under index r . It follows from the necessary condition proved above that $r = n$.

THEOREM 6. *If $n > 1$ and the compact continuum M is nearly homogeneous and does not contain n points such that M is aposyndetic at each of them with respect to each of the others, then M is indecomposable.*

This theorem follows from Theorems 2 and 5.

3. Homogeneous continua in the plane. The following definition is used in the proof of Theorem 7. A continuum M is said to have property W if every open subset of M contains both a point of a proper subset K of M and a point of $M - K$ such that K is an irreducible cutting⁷ of the plane.

THEOREM 7. *If the bounded plane continuum M is nearly homogeneous and separates the plane into a finite number of connected domains, then M is the boundary of each of its complementary domains.*

⁶ A continuum M is said to be *aposyndetic* at a point x of M with respect to a point y of M if there exists a subcontinuum K of M and an open subset U of M such that $M - y \supset K \supset U \supset x$.

⁷ A point set K is said to be an *irreducible cutting* of the plane if K separates the plane and no proper subset of K does so. Every such set is closed [20, page 43].

LEMMA 7.1. *If the bounded plane continuum M has property W and contains no domain, then every domain intersecting M intersects infinitely many complementary domains of M .*

PROOF OF LEMMA 7.1. Suppose that this lemma is not true. Then there exists a positive integer n such that (1) every domain intersecting M intersects at least n complementary domains of M and (2) some domain D intersects M and does not intersect $n+1$ complementary domains of M . There is a proper subset K of M such that D intersects both K and $M-K$ and K is an irreducible cutting of the plane. There is a connected domain D' lying in D and intersecting M but not K . There is a complementary domain E of K intersecting D but not D' . Hence some complementary domain E' of M lies in E and intersects D but not D' . Since D' intersects n complementary domains of M , then D intersects $n+1$ complementary domains of M . This involves a contradiction.

PROOF OF THEOREM 7. Suppose that some complementary domain of M does not have M as its boundary. Since the boundary of this domain separates the plane, it follows from two theorems proved by Kuratowski [13, Theorems 2 and 4] that some proper subcontinuum K of M is an irreducible cutting of the plane. Let D be any open subset of M and let x be a limit point of $M-K$ lying in K . There is a homeomorphism T of M onto itself that carries x into a point of D . It follows from a theorem proved by Brouwer [5, p. 425] that $T(K)$ is an irreducible cutting of the plane. Since $T(K)$ is a proper subset of M and $T(x)$ is a limit point of $M-T(K)$, it follows that M has property W . Since a plane domain is invariant under a homeomorphism [4], it follows that M contains no domain. Hence by Lemma 7.1, every domain intersecting M intersects infinitely many complementary domains of M . This is a contradiction since M has only a finite number of complementary domains.

THEOREM 8. *If the bounded plane continuum M is nearly homogeneous and has more than two but only a finite number of complementary domains, then M is indecomposable.⁸*

PROOF. Suppose that M is decomposable. By Theorem 7, M is the boundary of each of its complementary domains. It follows from an argument given by Kuratowski that M is indecomposable under index 2. (See [14, Theorem 3] and the argument thereof.) This is contrary to Theorem 2.

⁸ The author does not know whether there exists a continuum which satisfies the hypothesis of this theorem.

THEOREM 9. *If $n > 1$ and the bounded plane continuum M is nearly n -homogeneous and is not a continuous curve, then M is indecomposable⁹.*

INDICATION OF PROOF. Suppose that M is the finished sum of five continua M_1, M_2, \dots, M_5 . Let K_1, K_2, \dots, K_5 be circular domains such that (1) $\bar{K}_1, \bar{K}_2, \dots, \bar{K}_5$ are mutually exclusive, (2) for each i ($i \leq 5$), K_i intersects M_i , and (3) if $j \neq i$, \bar{K}_i does not intersect M_j . By [17, Theorem 47, p. 132], there exist two open subsets U_1 and U_2 of M , a continuum H , and a sequence of mutually exclusive continua H_1, H_2, H_3, \dots such that (1) U_1 and U_2 are mutually exclusive, (2) for each n , H_n intersects both U_1 and U_2 and is a component of $M - (U_1 + U_2)$, and (3) the sequence H_1, H_2, H_3, \dots converges to H . Since M is nearly n -homogeneous, there exists a homeomorphism T of M onto itself that carries some point x_1 of H into a point of K_1 and some point x_2 of H into a point of K_2 . There exist open subsets R_1 and R_2 of M containing x_1 and x_2 respectively such that $T(R_1) \subset K_1$ and $T(R_2) \subset K_2$. Since infinitely many of the sets H_1, H_2, H_3, \dots intersect both R_1 and R_2 , then infinitely many of the sets $T(H_1), T(H_2), T(H_3), \dots$ intersect both K_1 and K_2 . Consequently there exists an infinite sequence H'_1, H'_2, H'_3, \dots of mutually exclusive subcontinua of M such that (1) for each n , H'_n intersects both \bar{K}_1 and \bar{K}_2 and is a component of $M - M \cdot (K_1 + K_2)$, and (2) for each i ($i \leq 2$), some arc $p_i q_i$ on the boundary J_i of K_i contains the sets $J_i \cdot H'_1, J_i \cdot H'_2, J_i \cdot H'_3, \dots$ in the order named from p_i to q_i . For each n , there is an arc $a_n b_n$ irreducible from \bar{K}_1 to \bar{K}_2 such that (1) a_n is between $J_1 \cdot H'_n$ and $J_1 \cdot H'_{n+1}$ on $p_1 q_1$ and b_n is between $J_2 \cdot H'_n$ and $J_2 \cdot H'_{n+1}$ on $p_2 q_2$ and (2) $a_n b_n$ does not intersect M . For each n , let D_n denote the complementary domain of $M + \bar{K}_1 + \bar{K}_2$ which contains $a_n b_n - (a_n + b_n)$. Since the domains D_1, D_2, D_3, \dots are mutually exclusive, it can be shown by an argument quite similar to the one used in Case 1 of the proof of Theorem 9 of [6] that no one of the sets \bar{K}_3, \bar{K}_4 , and \bar{K}_5 intersects more than two of these domains. Hence one of the domains D_1, D_2, D_3, \dots does not intersect $\bar{K}_3 + \bar{K}_4 + \bar{K}_5$. From this it follows that if $i \neq j$ ($i, j \leq 5$), there is a complementary domain Z of $M + \bar{K}_i + \bar{K}_j$ and an arc t not intersecting M such that (1) Z does not intersect any of the sets $\bar{K}_1, \bar{K}_2, \dots, \bar{K}_5$ and (2) t is irreducible from \bar{K}_i to \bar{K}_j and lies, except for its end points, in Z . Let Z_1, Z_2, \dots, Z_{10} be ten such domains corresponding to the ten ways the integers i and j can be chosen. An argument similar to the one used in the proof of Theorem 5 of [7] can be used to show that some two of these ten domains have a point in common. Consider the case in which (1) Z_1 and Z_2 have a point in common, (2) Z_1 is a comple-

⁹ Bing [2, Theorem 14] has shown that such a continuum exists.

mentary domain of $M + \bar{K}_1 + \bar{K}_2$ and for each i ($i \leq 2$), $\bar{K}_i - M \cdot \bar{K}^i$ contains a boundary point of Z_1 , and (3) Z_2 is a complementary domain of $M + \bar{K}_3 + \bar{K}_r$, where $r \neq 3$, and for each i ($i = 3, r$), $\bar{K}_i - M \cdot \bar{K}^i$ contains a boundary point of Z_2 . Then Z_1 contains Z_2 together with every boundary point of Z_2 that belongs to $\bar{K}_3 - M \cdot \bar{K}_3$. Hence Z_1 intersects \bar{K}_3 . As this involves a contradiction, it follows that there is a positive integer k less than five such that M is indecomposable under index k . Since M is nearly homogeneous, then by Theorem 2, M is indecomposable.

THEOREM 10. *If $n > 1$ and the bounded plane continuum M is n -homogeneous, then M is a simple closed curve.*

PROOF. Suppose that M is indecomposable. Let x_1, x_2, \dots, x_n be n points lying in the same composant of M and let y_1, y_2, \dots, y_n be n points of M such that no composant of M contains all of them. Since some proper subcontinuum of M contains $x_1 + x_2 + \dots + x_n$ and no proper subcontinuum of M contains $y_1 + y_2 + \dots + y_n$, there does not exist a homeomorphism of M onto itself that carries $x_1 + x_2 + \dots + x_n$ into $y_1 + y_2 + \dots + y_n$. As this involves a contradiction, then M is decomposable. Since M is nearly n -homogeneous, it follows from Theorem 9 that M is a continuous curve. By Theorem 1, M is homogeneous. Mazurkiewicz [15] has shown that every homogeneous bounded plane continuous curve is a simple closed curve. Hence M is a simple closed curve.

4. Questions and remarks. So far as the author knows, the following two questions have not as yet been answered. (1) Does there exist a homogeneous bounded plane continuum which separates the plane and is not a simple closed curve? (See [12, p. 149] and [3, p. 49].) Cohen [9] has shown that if such a continuum exists, then it does not contain a simple closed curve. (2) Is every homogeneous decomposable bounded plane continuum M a simple closed curve? Jones [10] has shown that the answer is in the affirmative if M is aposyndetic.

There does exist a *nearly* homogeneous bounded plane dendron. Such a set can be easily constructed. The set of all end points of any such dendron M would be everywhere dense in M .

There exists a decomposable continuum which satisfies the hypothesis of Theorem 3. For each i ($i \leq 2$), let H_i be a set indicated by Fig. 1 of [6] such that the common part of H_1 and H_2 is the point O . Let K_i be a composant of H_i not containing O and let x_i be a point of K_i . The decomposable continuum $H_1 + H_2$ is irreducible between x_1 and x_2 and is nearly homogeneous over $x_1 + x_2$.

Anderson [1] has shown that there exists a one-dimensional compact continuous curve M such that for each n , M is strongly n -homogeneous. If $n > 3$, then no simple closed curve is strongly n -homogeneous. Hence it follows from Theorem 10 that if $n > 3$, then no bounded *plane* continuum is strongly n -homogeneous.

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