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A PROPERTY OF QUASI-CONFORMAL MAPPING

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Let W_1 and W_2 be two open Riemann surfaces such that there is a quasi-conformal [1] mapping h of W_1 onto W_2. Then it is known [2] that either both W_1 and W_2 have Green's function or else neither has, i.e., quasi-conformal mapping preserves the class O_G of those surfaces which have no Green's function. From this one is led to the conjecture that quasi-conformal mapping preserves the classes O_{HB} and O_{HD} of surfaces which have no bounded harmonic function or harmonic functions with a finite Dirichlet integral, respectively. In the present paper we shall show that the ring HD of bounded harmonic functions with a finite Dirichlet integral is the same for both W_1 and W_2. This has as a consequence not only the result of Pfluger on the preservation of the class O_G but also the preservation of the class O_{HD} under quasi-conformal mapping.

Whether or not O_{HB} is preserved under quasi-conformal mapping is an open question. Another interesting open question is whether or not two topologically equivalent surfaces which have the same ring HBD defined on them admit of a quasi-conformal mapping from one to the other.

1. Some rings of functions. Let $BD = BD_i$ be the ring of all piecewise smooth functions defined on the Riemann surface $W = W_i$ which are bounded and have a finite Dirichlet integral. A topology is introduced in BD by defining

$$f_\nu \to f$$

if $|f_\nu|$ is uniformly bounded, f converges to f uniformly on every

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compact subset of W, and $D[f_\nu - f]$ converges to zero, where $D[\ ]$ denotes the Dirichlet integral:

$$D[f] = \iint df * d\bar{f}.$$

A subset S of BD is then called closed if it contains all limits of sequences from S.

We let K be the ideal consisting of all those functions in BD which vanish outside some compact set and denote by $\overline{K}$ those functions which are limits (in BD) of sequences from K. It is shown in [**3**] that $\overline{K}$ is closed and that every function $f \in BD$ has a unique decomposition

$$f = f_K + u \tag{1}$$

where $f_K \in \overline{K}$ and $u \in HBD$, the space of harmonic functions in BD (with the convention that the constants belong to HBD only if W is hyperbolic). As a vector space HBD is isomorphic to the quotient space $BD/\overline{K}$ and we use this isomorphism to define [**4**] a multiplication in HBD.

If we let π be the homomorphism which takes f into the harmonic part u of its decomposition (1), then π is a ring homomorphism by the definition of multiplication in HBD. Moreover, π is continuous.

2. A class of linear functionals. Let L consist of those continuous linear functionals on BD which annihilate $\overline{K}$. Then L is a total set of linear functionals for the subspace HBD. For, if $u \in HBD$ and u is not constant, then

$$l(f) = \iint df * d\bar{u}$$

is continuous on BD and annihilates functions in $\overline{K}$. But

$$l(u) = D[u] > 0.$$

Also there is by Theorem 3 of [**3**] a linear functional $l \in L$ such that

$$l[1] \neq 0.$$

Now if $l \in L$, then

$$l[f] = l[(1 - \pi)f] + l[\pi f] = l(\pi f) \tag{2}$$

since $(1 - \pi)f \in \overline{K}$.

3. Mappings induced by h. Let now h be a quasi-conformal mapping of W_1 onto W_2. Then we let τ be the mapping of BD_1 onto BD_2 defined by

$$\tau f(P) = f[h^{-1}(P)], \qquad P \in W_2,$$

and we have

$$\tau^{-1} f(Q) = f[h(Q)], \qquad Q \in W_1.$$

Moreover, τ is a homeomorphism of BD_1 onto BD_2. For if $|f_\nu|$ is uniformly bounded so is $|\tau f_\nu|$; while if f_ν converges to f uniformly on compact subsets of W_1, then τf_ν converges to τf uniformly on compact subsets of W_2. Also,

$$D[\tau f_\nu - \tau f] \leqq kD[f - f_\nu],$$

where k is the bound on the dilation quotient of the mapping h. Hence τ is continuous. Similarly τ^{-1} is continuous and τ is a homeomorphism. Since τ is linear, it is also an isomorphism.

We define σ to be the adjoint of τ^{-1}, i.e., for a continuous linear functional l on BD_1 we define $l\sigma$ by

$$l\sigma[f] = l[\tau^{-1} f], \qquad f \in BD_2. \tag{3}$$

Then

$$l\sigma[\tau f] = l[f]. \tag{4}$$

Since τ^{-1} takes $\overline{K}_2$ into $\overline{K}_1$, σ takes L_1 into L_2.

4. **The principle theorem.** We first prove the following relations

$$\pi_1 \tau^{-1} \pi_2 \tau \pi_1 = \pi_1, \tag{5}$$

$$\pi_2 \tau \pi_1 \tau^{-1} \pi_2 = \pi_2. \tag{6}$$

For any $l \in L_1$ and $u \in HBD_1$, we have

$$l[\pi_1 \tau^{-1} \pi_2 \tau u] = l[\tau^{-1} \pi_2 \tau u]$$

by (2). By the definition of σ we may write this as

$$\begin{aligned} l[\pi_1 \tau^{-1} \pi_2 \tau u] &= l\sigma[\pi_2 \tau u] \\ &= l\sigma[\tau u] \end{aligned}$$

by (2) since $l\sigma \in L_2$. By (3) we have

$$l[\pi_1 \tau^{-1} \pi_2 \tau u] = l[u].$$

Since $\pi_1 \tau^{-1} \pi_2 \tau u \in HBD_1$ and L_1 is total for HBD_1, we must have

$$\pi_1 \tau^{-1} \pi_2 \tau u = u$$

whence (5) follows. Similarly for (6).

THEOREM 1. *The mapping $\pi_2 \tau$ is a homeomorphic isomorphism of the* *ring HBD_1 onto the ring HBD_2.*

PROOF. The mapping $\pi_2\tau$ is a ring homomorphism since

$$\pi_2\tau = \pi_2\tau\pi_1 \qquad \text{on } HBD_1,$$

and π_2, τ, and π_1 are all ring homomorphisms. Moreover, $\pi_2\tau$ is continuous since π_2 and τ are. So also is $\pi_1\tau^{-1}$. But by (5) and (6)

$$(\pi_2\tau)^{-1} = \pi_1\tau^{-1}$$

whence $\pi_2\tau$ is one-to-one onto and bicontinuous, proving the theorem.

Since $W \in O_G$ is equivalent to $HBD(W)$ empty and $W \in O_{HD}$ is equivalent to saying $HBD(W)$ has dimension less than two, we have the following corollaries:

COROLLARY 1. *The class O_G is preserved under quasi-conformal mapping.*

COROLLARY 2. *The class O_{HD} is preserved under quasi-conformal mapping.*

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