ON MAXIMIZING AN INTEGRAL WITH A SIDE CONDITION

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The problem to be discussed in this paper is that of finding an admissible function, $\hat{p}(x)$, which makes the integral

$$(1) \qquad \mathcal{J}(p) = \int_E F(p(x), x)dx$$

an *absolute maximum* subject to a side condition of the type

$$(2) \qquad \mathcal{G}(p) = \int_E G(p(x), x)dx = c.$$

The class, $\mathcal{P}$, of admissible functions is to include all $p(x)$ satisfying

A_1: $u(x) \leqq p(x) \leqq v(x)$,

A_2: $F(p(x), x)$ and $G(p(x), x)$ summable over E.

It will further be assumed that

H_1: E is a compact subset of the reals,

H_2: $u(x)$ and $v(x)$ continuous for $x \in E$,

H_3: $F(p, x)$ and $G(p, x)$ continuous for $x \in E$ and $u(x) \leqq p \leqq v(x)$,

H_4: $\inf_{p \in \mathcal{P}} \mathcal{G}(p) < c < \sup_{p \in \mathcal{P}} \mathcal{G}(p)$.

Under these conditions the problem can often be solved by forming

$$(3) \qquad h(\theta, x) = \operatorname*{Max}_{u(x) \leqq \mu \leqq v(x)} h(\theta, x, \mu)$$

where

$$(4) \qquad h(\theta, x, \mu) = \cos \theta F(\mu, x) + \sin \theta G(\mu, x).$$

If $\theta = \theta_c$ in the open interval $(-\pi/2, \pi/2)$ and an admissible function $\mu_{\theta_c}(x)$ can be found such that $\mu_{\theta_c}(x)$ maximizes $h(\theta_c, x, \mu)$ for each x, and if

$$(5) \qquad \mathcal{G}(\mu_{\theta_c}) = c,$$

this function, $\mu_{\theta_c}(x)$, will be a solution to the problem. This follows because for all $x \in E$ and admissible $p(x)$

$$(6) \qquad h(\theta_c, x, \mu_{\theta_c}(x)) \geqq h(\theta_c, x, p(x)).$$

On integrating,

$$(7) \qquad \cos \theta_c \mathcal{J}(\mu_{\theta_c}) + \sin \theta_c \mathcal{G}(\mu_{\theta_c}) \geqq \cos \theta_c \mathcal{J}(p) + \sin \theta_c \mathcal{G}(p)$$

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so that for all $p(x)$ satisfying the side condition (2)

$$\mathcal{J}(\mu_{\theta_c}) \geqq \mathcal{J}(p). \tag{8}$$

Thus, the problem is solved if the existence of θ_c can be demonstrated. Assurance of this is given by the following theorem, and in the course of its proof an admissible $\mu_{\theta_c}(x)$ is constructed:

THEOREM. *If $\mathcal{J}(p)$ and $\mathcal{G}(p)$ are defined as in* (1) *and* (2), *if the class $\mathcal{P}$ is defined by* A_1 *and* A_2, *and if hypotheses* H_1, H_2, H_3, *and* H_4 *are all satisfied, then there exists θ_c in the open interval* $(-\pi/2, \pi/2)$ *and* $\mu_{\theta_c}(x) \in \mathcal{P}$ *such that for all* $x \in E$

$$h(\theta_c, x, \mu_{\theta_c}(x)) = \operatorname*{Max}_{u(x) \leqq \mu \leqq v(x)} h(\theta_c, x, \mu) \tag{9}$$

and

$$\mathcal{G}(\mu_{\theta_c}) = c, \tag{10}$$

and, for all $p \in \mathcal{P}$ *such that* $\mathcal{G}(p) = c$,

$$\mathcal{J}(\mu_{\theta_c}) \geqq \mathcal{J}(p). \tag{11}$$

The final assertion (11) has already been proved. In connection with proving the existence of θ_c we first establish the continuity of $h(\theta, x)$.

Define $F(p, x) = F(v(x), x)$ for $p > v(x)$ and $F(p, x) = F(u(x), x)$ for $p < u(x)$, similarly for $G(p, x)$. $h(\theta, x)$ is then the maximum of $h(\theta, x, \mu)$ over the closed interval $\inf_{x \in E} u(x) \leqq \mu \leqq \sup_{x \in E} v(x)$. As the upper envelope of the continuous functions $h(\theta, x, \mu)$, $h(\theta, x)$ is lower semi-continuous. To show that $h(\theta, x)$ is also upper semi-continuous consider the set (θ, x, μ) for which $h(\theta, x, \mu) \geqq \rho$. This set is bounded and closed. Its projection onto the θ, x domain is therefore also closed. This projection is, however, the set of points (θ, x) for which $h(\theta, x) \geqq \rho$. Consequently, $h(\theta, x)$ is also upper semi-continuous.

We now consider for each fixed θ and x the point set $\{\mu_\theta^\alpha(x)\}$ of maximizing μ's. This set is closed by virtue of the continuity of $h(\theta, x, \mu)$ in μ, and bounded by $u(x)$ and $v(x)$. Therefore, there exist functions $\mu_\theta^+(x)$ and $\mu_\theta^-(x)$ in $\{\mu_\theta^\alpha(x)\}$ satisfying

$$G(\mu_\theta^+(x), x) = \sup_\alpha G(\mu_\theta^\alpha(x), x), \tag{12}$$

$$G(\mu_\theta^-(x), x) = \inf_\alpha G(\mu_\theta^\alpha(x), x). \tag{13}$$

We shall now show that $G(\mu_\theta^+(x), x)$ is upper semi-continuous and $G(\mu_\theta^-(x), x)$ is lower semi-continuous. Let $\{\theta_j, x_j\}$ be any sequence of points in the θ, x domain tending to (θ_0, x_0) for which $\mu^+_{\theta_j}(x_j)$ converges. Then,

$$
\begin{aligned}
h(\theta_0, x_0, \lim_{j\to\infty} \overset{+}{\mu}_{\theta_j}(x_j)) &= \lim_{j\to\infty} h(\theta_j, x_j, \overset{+}{\mu}_{\theta_j}(x_j)) \\
&= \lim_{j\to\infty} h(\theta_j, x_j) \\
&= h(\theta_0, x_0)
\end{aligned} \tag{14}
$$

so that

$$
\lim_{j\to\infty} \overset{+}{\mu}_{\theta_j}(x_j) = \overset{\alpha}{\mu}_{\theta_0}(x_0) \tag{15}
$$

for some α. Consequently,

$$
\begin{aligned}
\lim_{j\to\infty} G(\overset{+}{\mu}_{\theta}\cdot(x_j), x_j) &= G(\lim_{j\to\infty} \overset{+}{\mu}_{\theta_j}(x_j), x_0) \\
&= G(\overset{\alpha}{\mu}_{\theta_0}(x_0), x_0) \\
&\leqq G(\overset{+}{\mu}_{\theta_0}(x_0), x_0).
\end{aligned} \tag{16}
$$

Similarly, $G(\mu_\theta^-(x), x)$ is lower semi-continuous.

These functions are therefore summable over E, and since $F(\mu_\theta^+(x), x)$ and $F(\mu_\theta^-(x), x)$ can be expressed in terms of the G's together with $h(\theta, x)$, they too are summable over E. Thus μ_θ^+ and μ_θ^- are admissible functions.

Furthermore, the integral $\mathcal{G}(\mu_\theta^+)$ is upper semi-continuous and the integral $\mathcal{G}(\mu_\theta^-)$ is lower semi-continuous. This follows via Fatou's lemma, viz.

$$
\begin{aligned}
\limsup_{\theta\to\theta_0} \int_E G(\overset{+}{\mu}_\theta(x), x)dx &\leqq \int_E \limsup_{\theta\to\theta_0} G(\overset{+}{\mu}_\theta(x), x)dx \\
&\leqq \int_E G(\overset{+}{\mu}_{\theta_0}(x), x)dx.
\end{aligned} \tag{17}
$$

Similarly, $\mathcal{G}(\mu_\theta^-)$ is lower semi-continuous.

We now define the sets S^+ and S^- as follows:

$$
S^+ = \{\theta \mid -\pi/2 < \theta < \pi/2 \text{ and } \mathcal{G}(\overset{+}{\mu}_\theta) \geqq c\}, \tag{18}
$$

$$
S^- = \{\theta \mid -\pi/2 < \theta < \pi/2 \text{ and } \mathcal{G}(\overset{-}{\mu}_\theta) \leqq c\}. \tag{19}
$$

By virtue of the semi-continuity of the integrals these sets are both closed relative to the open interval $(-\pi/2, \pi/2)$. Neither exhausts $(-\pi/2, \pi/2)$. This follows since for all $p\in\mathcal{P}$

(20) $$\mathcal{G}(\overset{+}{\mu}_{-\pi/2}) \leqq \mathcal{G}(p)$$

so that

(21) $$\mathcal{G}(\overset{+}{\mu}_{-\pi/2}) < c.$$

But

(22) $$\limsup_{\theta\to -\pi/2} \mathcal{G}(\overset{+}{\mu_\theta}) \leqq \mathcal{G}(\overset{+}{\mu}_{-\pi/2}) < c$$

so that there exists $\theta \in (-\pi/2, \pi/2)$ such that $\mathcal{G}(\mu_\theta^+) < c$. Thus S^+ is not equal to $(-\pi/2, \pi/2)$. Similarly S^- is not equal to $(-\pi/2, \pi/2)$.

Since $(-\pi/2, \pi/2)$ is connected, either

(23) $$S^+ \cap S^- \neq 0$$

or

(24) $$\operatorname{comp} S^+ \cap \operatorname{comp} S^- \neq 0,$$

otherwise S^+ and S^- would form a separation. But since $\mathcal{G}(\mu_\theta{}^+) \geqq \mathcal{G}(\mu_\theta^-)$, the second alternative is absurd. We therefore pick $\theta = \theta_c$ in $S^+ \cap S^-$. Letting

(25) $$\mu_\lambda(x) = \begin{cases} \overset{+}{\mu}_{\theta_c}(x) & \text{for} \quad x \leqq \lambda, \\ \overset{-}{\mu}_{\theta_c}(x) & \text{for} \quad x > \lambda, \end{cases} \qquad \lambda \in E,$$

we consider the integral

$$\mathcal{G}(\mu_\lambda) = \int_{x \leqq \lambda,\, x \in E} G(\overset{+}{\mu}_{\theta_c}(x), x)\,dx + \int_{x > \lambda,\, x \in E} G(\overset{-}{\mu}_{\theta_c}(x), x)\,dx.$$

This integral is a continuous function of λ and covers the range

$$\mathcal{G}(\overset{-}{\mu}_{\theta_c}) \leqq c \leqq \mathcal{G}(\overset{+}{\mu}_{\theta_c}).$$

Thus, λ_c exists for which $\mathcal{G}(\mu_{\lambda_c}) = c$. Therefore, $\mu_{\lambda_c}(x) = \mu_{\theta_c}(x)$ is a solution, and the theorem is proved.

The theorem above provides the basis for a maximization procedure which may be stated as follows:

In order to determine a maximizing function $\hat{p} \in \mathcal{P}$ *for the integral* (1) *with side condition* (2), H_1, H_2, H_3, *and* H_4 *being assumed, it is necessary only to maximize the associated integrand*

$$\cos\theta F(p(x), x) + \sin\theta G(p(x), x)$$

over the admissible values of $p(x)$ *for almost all* $x \in E$ *and choose* θ *so that condition* (2) *is satisfied with the corresponding maximizing* p.

Through establishing the existence of θ_c the theorem asserts that the above procedure will always yield a solution. Consequently,

COROLLARY 1. *The problem has at least one solution.*

Furthermore,

COROLLARY 2. *Every solution p^* conforms to the maximizing procedure—and with common θ.*

Assuming the denial, we put $\theta=\theta_c$ and have

$$\cos \theta_c F(p^*(x), x) + \sin \theta_c G(p^*(x), x) < \cos \theta_c F(\mu_{\theta_c}(x), x) + \sin \theta_c G(\mu_{\theta_c}(x), x)$$

on some subset of E of measure >0. An integration leads to the contradiction.

Hypothesis H_2 may be somewhat revised, as indicated by

COROLLARY 3. *Bound free problems (i.e., $u(x)=-\infty$ and $v(x)=+\infty$), as well as problems with discontinuous bound functions, conform to the maximization procedure provided it is known that a bounded solution exists.*

COROLLARY 4. *If the bound free problem has a solution p', then p' conforms to the maximization procedure on any compact subset $E' \subseteq E$ on which p' is bounded.*

Remarks. Regarding the admissibility conditions for the class $\mathcal{P}$, the first, A_1, was imposed in order to bring problems demanding such bounds on p into the domain of the maximizing principle. It may be relaxed under certain circumstances depending on the form of F and G (for example, see Corollaries 3 and 4). A_2 is obviously needed. A third, A_3, imposing measurability on all $p \in \mathcal{P}$ might possibly be included if a suitable rule could be set down (such as taking sup or inf perhaps) for selecting the $\mu_\theta^+(x)$ and $\mu_\theta^-(x)$ at each point.

Although hypotheses H_1 and H_3 are about as weak as they need be for many practical problems, they might still be further weakened. This has not been investigated. On the other hand, H_4 is necessary, for, consider the case $c = \text{Max } \mathcal{G}(p)$. Here a permissible θ is $\pi/2$, and the maximizing procedure yields all $p \in \mathcal{P}$ satisfying $\mathcal{G}(p)=c$. Some of these may yield $\mathcal{J}(p)$ larger than others.

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