TOPOLOGICAL DEGREE OF SOME MAPPINGS

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1. In previous papers [2; 3], the problem of proving an existence theorem for a certain functional equation was reduced to that of computing the topological degree of a mapping in Euclidean n-space defined by homogeneous polynomials or infinite series. The complex case of the latter problem was solved in [4]. Since the problem is analogous to that of studying the roots of a polynomial equation, we would expect the real case to be more complicated. Here we obtain a result that is an analogue of the theorem that a real polynomial equation of odd degree has at least one real root. Also we describe the solution for the case $n=2$ if the mapping is defined by homogeneous polynomials.

2. We consider the mapping of real Euclidean n-space R^n into itself,

$$M:(x_1, \cdots, x_n) \to (x_1', \cdots, x_n')$$

defined by

$$x_i' = \sum_{m=2}^{\infty} \sum_j a_{j_1 \cdots j_n} x_1^{j_1} \cdots x_n^{j_n} \qquad (i = 1, \cdots, n)$$

where $\sum_j$ denotes a summation taken over all sets of non-negative integers $j_1, \cdots, j_n$ such that $\sum_{q=1}^n j_q = m$. The problem is to determine the topological degree at 0 of M. Let $\mathcal{M}$ be the mapping of complex Euclidean $\mathcal{R}^n$ into itself that corresponds to M, i.e., $\mathcal{M}$ is defined by

$$z_i' = \sum_{m=2}^{\infty} \sum_j a_{j_1 \cdots j_n} z_1^{j_1} \cdots z_n^{j_n} \qquad (i = 1, \cdots, n).$$

Let S be a sphere in R^n with center 0 such that $d[M, S, 0]$, the topological degree at 0 of M relative to S, is defined, and let $\mathcal{S}$ be the corresponding sphere in $\mathcal{R}^n$, i.e., a sphere in $\mathcal{R}^n$ with center 0 and radius equal to the radius of S. Suppose first that $d[\mathcal{M}, \mathcal{S}, 0]$ is defined, i.e., suppose $\mathcal{M} \neq 0$ on the surface of $\mathcal{S}$. We prove:

$$|d[M, S, 0]| \leqq d[\mathcal{M}, \mathcal{S}, 0] \tag{1}$$

and

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$$d[M, S, 0] \equiv d[\mathfrak{M}, \mathfrak{S}, 0] \pmod{2}. \tag{2}$$

By Lemma 3.1 of [3] (which is a special case of a theorem due to Sard [6]) and the fundamental property of topological degree [1, Deformationssatz, p. 424] there is a real point p near 0 such that

(a) $$d[M, S, 0] = d[M, S, p],$$

(b) $$d[\mathfrak{M}, \mathfrak{S}, 0] = d[\mathfrak{M}, \mathfrak{S}, p],$$

(c) The set $M^{-1}(p)$ is finite.

Let $q_1, \cdots, q_r$ be the elements of $M^{-1}(p)$ and let J be the Jacobian of M. Then

$$d[M, S, p] = \sum_{i=1}^{r} \operatorname{sign} J(q_i).$$

Since $J(q_i) \neq 0$ for $i=1, \cdots, n$, the points $q_1, \cdots, q_r$ are isolated points in the set $\mathfrak{M}^{-1}(p)$. As proved in [4], the topological index of $\mathfrak{M}$ at each q_i is $+1$. Hence from the properties of topological degree [1, Satz 11, p. 472], it follows that

$$d[\mathfrak{M}, \mathfrak{S}, p] = d[\mathfrak{M}, \mathfrak{S} - S, p] + r.$$

But since the coefficients in $\mathfrak{M}$ are real, it follows easily that $d(\mathfrak{M}, \mathfrak{S} - S, p)$ is a positive, even number. (This is proved in [5].) Since

$$d[M, S, p] \equiv r \pmod{2},$$

the proof is complete.

Now suppose $\mathfrak{M}=0$ at some point on the surface of $\mathfrak{S}$. We assume that the coefficients in the series that define M and $\mathfrak{M}$ may be varied slightly so that the following result is obtained: the mappings M_1 and $\mathfrak{M}_1$ in R^n and $\mathfrak{R}^n$, respectively, defined by the new series are such that

(α) $d[M, S, 0] = d[M_1, S, 0]$,

(β) $\mathfrak{M}_1$ is different from zero on $\mathfrak{S}$, i.e., $d[\mathfrak{M}_1, \mathfrak{S}, 0]$ is defined.

Then applying the preceding argument to M_1 and $\mathfrak{M}_1$, we obtain:

$$\left| d[M, S, 0] \right| = \left| d[M_1, S, 0] \right| \leqq d[\mathfrak{M}_1, \mathfrak{S}, 0] \tag{3}$$

and

$$d[M, S, 0] = d[M_1, S, 0] \equiv d[\mathfrak{M}_1, \mathfrak{S}, 0] \pmod{2}. \tag{4}$$

In particular, if M is defined by homogeneous polynomials, i.e.,

$$x_i' = P_{k_i}(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n)$$

where P_{k_i} is homogeneous of degree k_i in $x_1, \cdots, x_n$, then

$$|d[M, S, 0]| \leqq \prod_{i=1}^{n} k_i \tag{5}$$

and

$$d[M, S, 0] \equiv \prod_{i=1}^{n} k_i \pmod 2. \tag{6}$$

This follows from the preceding paragraphs, and the fact, proved in [4], that the topological index of the mapping in complex Euclidean n-space defined by

$$z_i' = P_{k_i}(z_1, \cdots, z_n) \qquad (i = 1, \cdots, n)$$

is $\prod_{i=1}^{n} k_i$.

Results (5) and (6) may also be obtained by using Bezout's Theorem. This was pointed out to me by R. Brauer before the proof given here was obtained.

3. If $n=2$ and the mapping is defined by homogeneous polynomials, a solution of the problem can easily be given. First by varying the coefficients slightly (so slightly that the topological degree relative to the unit circle is not affected) we obtain a mapping M defined by

$$x_1' = P_{k_1}(x_1, x_2) = C_1 \prod_{i=1}^{n} (x_1 - \alpha_i x_2)^{p_i},$$

$$x_2' = P_{k_2}(x_1, x_2) = C_2 \prod_{j=1}^{m} (x_1 - \beta_j x_2)^{q_j},$$

where C_1, C_2 are real constants. Since C_1 and C_2 affect only the sign of the topological degree, we may disregard them. The topological degree can be determined by investigating the changes of sign of P_{k_1} and P_{k_2} as (x_1, x_2) varies over the boundary of the unit circle. Consequently we may disregard factors $(x_1 - \alpha_i x_2)$ or $(x_1 - \beta_j x_2)$ which appear with even exponents or in which α_i and β_j are complex since none of these contributes to a change of sign of P_{k_1} or P_{k_2}. So we are left with real factors all having exponent one.

Now if there is a pair α_i, α_{i+1} ($\alpha_i < \alpha_{i+1}$) such that no β_j lies between then (i.e., there is no β_j such that $\alpha_i < \beta_j < \alpha_{i+1}$) then the factors $(x_1 - \alpha_1 x_2)$ and $(x_1 - \alpha_{i+1} x_2)$ may be disregarded since they contribute no significant change of sign to P_{k_1}. Similarly for pairs

β_i, β_{i+1}. Finally if α_r and α_s are the smallest and largest of the array of numbers $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_m$, then the factors $(x_1-\alpha_r x_2)$ and $(x_1-\alpha_s x_2)$ may be disregarded. Similarly, $(x_1-\beta_r x_2)$ and $(x_1-\beta_s x_2)$ may be disregarded if β_r and β_s are the smallest and largest.

Now if there are no remaining factors in P_{k_1} or in P_{k_2}, the topological degree is zero. Otherwise there remain factors containing numbers $\alpha_1, \cdots, \alpha_w$ and $\beta_1, \cdots, \beta_w$ where all the α's and β's are distinct and, if the subscript labelling is according to magnitude,

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_w < \beta_w$$

or

$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_w < \alpha_w.$$

In the first case the degree is w; in the second case $-w$.

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