AN ALTERNATIVE PROOF OF A THEOREM OF BECKENBACH[1]

EDGAR REICH

Beckenbach [1] has shown that the following analogue of Schwarz's lemma holds:

THEOREM. *If $f(z)$ is regular for $|z|<1$, and*

$$(1) \qquad I(\rho,\theta)=\int_0^\rho |f(re^{i\theta})|\,dr \leqq 1, \qquad 0\leqq \rho<1,\ 0\leqq\theta<2\pi,$$

then $I(\rho,\theta)\leqq\rho$. Equality for any (ρ_0,θ_0), with $\rho_0>0$, implies $f(z)=e^{i\alpha}$, α real.

We shall derive this theorem by using a device employed by Landau [2] for proving a theorem of Hardy.

By a change of variable of integration, (1) becomes

$$(2) \qquad I(\rho,\theta)=\rho\int_0^1 |f(\rho te^{i\theta})|\,dt \leqq 1, \qquad 0\leqq \rho<1,\ 0\leqq\theta<2\pi.$$

Consider the function

$$F(\rho,\theta,z)=z\int_0^1 f(zte^{i\theta})\exp\left[-i\arg f(\rho te^{i\theta})\right]dt,$$

$$0\leqq\rho<1,\ 0\leqq\theta<2\pi,\ 0\leqq|z|<1.$$

For every fixed ρ and θ, $0\leqq\rho<1$, $0\leqq\theta<2\pi$, F is an analytic function of z for $|z|<1$, vanishing for $z=0$. Let $\sigma=\theta+\arg z$. Then

$$|F(\rho,\theta,z)|\leqq|z|\int_0^1|f(zte^{i\theta})|\,dt=|z|\int_0^1|f(|z|te^{i\sigma})|\,dt\leqq 1$$

by (2). Schwarz's lemma now implies that

$$|F(\rho,\theta,z)|\leqq|z|.$$

In particular, for $z=\rho$, we have

$$|F(\rho,\theta,\rho)|=F(\rho,\theta,\rho)=I(\rho,\theta)\leqq\rho,$$

Received by the editors October 8, 1953 and, in revised form, January 18, 1954.

[1] The author wishes to thank Professor Beckenbach for suggesting the desirability of finding a purely complex-variable proof of this result which had previously been derived only by subharmonic function methods [1].

as was to be shown. The discussion for the case of equality in the conclusion of the theorem is straightforward.

More generally, by using a stronger form of Schwarz's lemma one can show that if in the hypothesis of the theorem one adds the statement that $f(z)$ has a zero of order n at the origin, then one can actually conclude that $I(\rho, \theta) \leqq \rho^n$.

References

1. E. F. Beckenbach, *A relative of the lemma of Schwarz*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 698–707.

2. E. Landau, *Neuer Beweis eines Hardyschen Satzes*, Arch. d. Math. u. Physik (3) vol. 25 (1916) pp. 173–178.

The Rand Corporation and
University of California, Los Angeles