NOTE ON FUNCTIONS BOUNDED AND ANALYTIC IN THE UNIT CIRCLE[1]

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W. Gross [1] was the first to give an example of an entire function $w=\phi(\zeta)$ with the property that, for each complex number w, there exists an uncountable set of asymptotic paths along which $\phi(\zeta)$ tends to w. We shall consider, in this note, the corresponding problem for functions analytic in the unit circle, U: $|z|<1$. Let $\phi(\zeta)$ be the entire function constructed by Gross, and let L be one of the paths in the set of asymptotic paths corresponding to an asymptotic value α. Let us cut the ζ-plane along L and map the slit region conformally onto the unit circle U by means of the function $\zeta=\zeta(z)$. The composed function $\phi(\zeta(z))$ is analytic in $|z|<1$, and, at the point z_0 on $|z|=1$ corresponding to $\zeta=\infty$, has the property that every complex value is an asymptotic value of $\phi(\zeta(z))$ on an uncountable set of paths terminating at z_0. However, if we assume a function defined in U to be bounded, it has at most one asymptotic value at every point of $|z|=1$ and tends to this asymptotic value radially if this exists. Thus it is natural to ask whether there exists a bounded analytic function in $|z|<1$ for which the set of radial limits at a set of linear measure zero on $|z|=1$ fills an open set.

THEOREM. *There exists a function $w=f(z)$, bounded and analytic in $|z|<1$, such that $f(e^{i\theta})=\lim_{r\to 1} f(re^{i\theta})$ has modulus 1 for almost all $e^{i\theta}$ on $|z|=1$, and such that, for each α with $|\alpha|\leqq 1$, the equation $f(e^{i\theta})=\alpha$ is satisfied on an uncountable set of $e^{i\theta}$ on $|z|=1$.*

We first divide $|w|\leqq 1$ into four parts:

$$A(1): 0 \leqq \psi \leqq \pi/2; \quad A(2): \pi/2 \leqq \psi \leqq \pi; \quad A(3): \pi \leqq \psi \leqq 3\pi/2;$$
$$A(4): 3\pi/2 \leqq \psi \leqq 2\pi \qquad (w = \rho e^{i\psi}),$$

and draw two disjoint slits $s(i)$ and $s(i')$ in the interior of each $A(i)$ $(1\leqq i\leqq 4)$. Next we divide $A(1)$ into four parts:

$$A(1, 1):\ 0 \leqq \rho \leqq 1/2, \quad 0 \leqq \psi \leqq \pi/4;$$
$$A(1, 2):\ 1/2 \leqq \rho \leqq 1, \quad 0 \leqq \psi \leqq \pi/4;$$
$$A(1, 3):\ 0 \leqq \rho \leqq 1/2, \ \pi/4 \leqq \psi \leqq \pi/2;$$

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$$A(1, 4):\ 1/2 \leqq \rho \leqq 1,\ \pi/4 \leqq \psi \leqq \pi/2,$$

and draw two disjoint slits $s(1, j)$ and $s(1, j')$, disjoint also from $s(1)$ and $s(1')$, in the interior of each $A(1, j)$ $(1 \leqq j \leqq 4)$. We do the same thing for $A(2)$, $A(3)$, and $A(4)$. Repeating the division into four parts we obtain sectors $\{A(i, j, \cdots, k)\}$ and slits $\{s(i, j, \cdots, k)\}$ and $\{s(i, j, \cdots, k')\}$. We denote by $F(i, j, \cdots, k)$ (and by $F(i, j, \cdots, k')$) the extended w-plane slit along the nine slits: $s(i, j, \cdots, k)$ ($s(i, j, \cdots, k')$ resp.), $\{s(i, j, \cdots, k, l)\}$, and $\{s(i, j, \cdots, k, l')\}$ $(l=1, 2, 3, 4)$. ($F(0)$ is the one slit along $\{s(i)\}$ and $\{s(i')\}$ $(i=1, 2, 3, 4)$.)

We now connect $F(0)$ with $F(1), F(1'), F(2), \cdots, F(4')$ along $s(1)$, $s(1')$, $s(2), \cdots, s(4')$ so that we obtain branch points at the end points of $s(i)$ and $s(i')$ $(1 \leqq i \leqq 4)$, and denote the surface thus obtained by G_1. Next connect G_1 with $\{F(i, j)\}$ and $\{F(i, j')\}$ $(i, j=1, 2, 3, 4)$ (we take two replicas of each of these F) along 64 boundary slits of G_1 and denote the resulting surface by G_2. We continue this procedure and obtain a surface G_n at each step. If we denote the surface $\bigcup_n G_n$ by G, $\{G_n\}$ forms an exhaustion of G.

We shall show that, if we take the lengths of the slits sufficiently small, G has a null boundary. We draw a disc in $F(0)$, say $|w| \geqq 1$, exclude it from G_n, and denote the remaining domain by G_n'. We denote by $\omega_n(P_0)$ the harmonic measure, at a fixed point P_0 of G_1', of the boundary slits of G_n with respect to the domain G_n'. If all the boundary slits of G_n reduce to points, G_n has obviously a null boundary. Therefore if we choose the lengths of these slits sufficiently small, $\omega_n(P_0)$ is less than $1/n$. Suppose that all slits are chosen in this way. Then $\omega_n(P_0) \to 0$ as $n \to \infty$. This means that G has a null boundary since $\{G_n\}$ is its exhaustion.

It is easily seen that G is of planar character. We can map G onto a domain D in the W-plane conformally in a one-to-one manner. Then, since G has a null boundary, D is bounded by a closed set E of capacity zero. We cut out that part of G lying over $|w| \geqq 1$ and denote the remaining connected surface over $|w| < 1$ by R. This corresponds to a subdomain D' of D, bounded by E and a countable number of closed curves $\{\Gamma_\nu\}$ each of which is an image of $|w| = 1$. We map the universal covering surface of D' onto U: $|z| < 1$ and denote the corresponding function which maps U onto D' by $W(z)$. Then by [2, p. 198], $W(z)$ tends to a point of E radially only on a set of linear measure zero on C. Therefore almost all radial limits of $W(z)$ lie on $\{\Gamma_\nu\}$. Consequently, if we compose the function $W(z)$ with the transformation $W \to w$ and if we denote the composed function by $f(z)$, then $f(z)$ has almost everywhere a radial limit of modulus one.

We shall show finally that, for any point α with $|\alpha| \leqq 1$, there are points of the power of the continuum on C at which $f(z)$ tends to α radially. We select a nested sequence $A(i) \supset A(i, j) \supset \cdots \to \alpha$. We start from the origin of the first sheet of R (a part of $F(0)$), enter into the second sheet of R by crossing over the slit $s(i)$ or $s(i')$, enter into the third sheet through $s(i, j)$ or $s(i, j')$, and so on. It is possible to limit the path on the Riemann surface in such a way that, as we move from one slit to another, the w-projection of the path converges to the point α. Since we may obtain nonhomotopic paths in an uncountable number of ways by choosing i or i', j or j', etc, we obtain a noncountable set of paths terminating at points of $|z| = 1$ such that $f(z) \to \alpha$ along any one of these paths. Then by Lindelöf's theorem, $f(z)$ tends radially to α along a set of radii of the power of the continuum. Hence the theorem is proved.

Bibliography

1. W. Gross, *Eine ganze Funktion, für die jede komplexe Zahl Konvergenzwert ist*, Math. Ann. vol. 79 (1918) pp. 201–208.

2. R. Nevanlinna, *Eindeutige analytische Funktionen*, Berlin, 1936.

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