$F(G) = V_T$, there exists an element y of G such that $Fy = x'$. Thus the distinct elements x, y of H are mapped by F into the same element x' of V_T, contrary to condition (ii).

References

1. Reinhold Baer, *The subgroup of elements of finite order of an abelian group*, Ann. of Math. vol. 37 (1936) pp. 766–781.
2. Hans Hahn, *Über die nichtarchimedischen Grössensysteme*, Sitz. der K. Akad. der Wiss., Math. Nat. Kl. vol. 116, Abt. IIa (1907) pp. 601–655.
3. M. Hausner and J. G. Wendel, *Ordered vector spaces*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 977–982.
4. Paul F. Conrad, *Embedding theorems for abelian groups with valuations*, Amer. J. Math. vol. 75 (1953) pp. 1–29. (Added in proof.)

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A NOTE ON A RECENT RESULT IN SUMMABILITY THEORY

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In a recent paper[1] by A. Mary Tropper, the following theorem is given:

Theorem. *In order that the regular[2] normal[3] matrix A shall sum a bounded divergent sequence, it is sufficient that*

(a) *its unique reciprocal B shall not be regular and*

(b) *there exists a normal matrix Q with[4] $\|Q\| < \infty$ whose columns are all null sequences, such that the matrix $C = BQ$ has bounded columns and $\|C\| = \infty$.*

The author points out that (a) is also a necessary condition, but does not prove the necessity of condition (b); the object of this note is to prove that condition (b) is necessary.

The proof given of the theorem quoted above holds if the K_r

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[1] Proc. Amer. Math. Soc. vol. 4 (1953) pp. 671–677.

[2] A regular matrix is one which satisfies $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}x_k = \lim_{k\to\infty} x_k$, whenever the latter limit exists.

[3] A normal matrix is a lower-semi matrix which has no zero elements in the leading diagonal. Such a matrix has a unique two-sided reciprocal; see e.g., R. G. Cooke, *Infinite matrices and sequence spaces*, Macmillan, 1950, p. 19.

[4] For an infinite matrix M, $\|M\| \equiv \sup_n \sum_k |m_{n,k}|$ is called the K_r bound of the matrix; a matrix whose K_r bound is finite is called a K_r matrix (ibid. pp. 25, 29).

matrix Q of condition (b) is required to be a lower-semi, but not necessarily a normal, matrix; the necessity of condition (b) will first be proved under these conditions, and then it will be shown that the existence of a *normal* K_r matrix Q is also necessary.

Suppose, therefore, that A sums the bounded divergent sequence $\{x_i\}$ to x. Without loss of generality, we may suppose $x=0$, since, if $x\neq 0$, A sums the bounded divergent sequence $\{x_i-x\}$ to 0.

Let $C^{(1)}$ be the normal matrix defined by

$$c_{i,j}^{(1)} = x_i \ (j < i); \qquad c_{i,i}^{(1)} = 1; \qquad c_{i,j}^{(1)} = 0 \ (j > i).$$

Any given column of $C^{(1)}$ is thus the sum of $\{x_i\}$ and a null sequence; hence its A-transform tends to 0, i.e., the columns of $Q^{(1)}\equiv AC^{(1)}$ all form null sequences.

Also,

$$(1) \quad \left| q_{r,s}^{(1)} \right| = \left| \sum_{t=1}^{r} a_{r,t} c_{t,s}^{(1)} \right| \leqq \sup_t \left| c_{t,s}^{(1)} \right| \sup_r \sum_{t=1}^{r} \left| a_{r,t} \right| < K \qquad \text{for all } r, s.$$

Now a diagonal matrix D, whose diagonal elements are all either 0 or 1, can be chosen in such a way that D contains an infinity of 1's and $Q=Q^{(1)}D$ is a K_r matrix. For, to choose such a matrix, let the values of n for which $d_n=1$ be denoted by $n_1, n_2, \cdots$, and suppose that $n_1, n_2, \cdots, n_p$ have been chosen such that

$$(2) \quad \sum_{s=1}^{n_\nu} \left| q_{r,s} \right| = \sum_{k=1}^{\nu} \left| q_{r,n_k}^{(1)} \right| \leqq 2K \text{ whenever } r \geqq n_\nu \ (\nu = 1, 2, \cdots, p).$$

Since the columns of $Q^{(1)}$ tend to 0, we may choose $n_{p+1}>n_p$ such that

$$\sum_{k=1}^{p} \left| q_{r,n_k}^{(1)} \right| \leqq K \qquad (r \geqq n_{p+1}),$$

and hence, by (1),

$$\sum_{s=1}^{n_{p+1}} \left| q_{r,s} \right| \leqq 2K \qquad (r \geqq n_{p+1}).$$

We may begin this construction by taking $n_1=1$, and it continues indefinitely, so that D contains an infinity of 1's.

We now show that with D so defined, $\|Q\| < \infty$. Given any value of r, there exists p such that $n_p \leqq r < n_{p+1}$; then

$$\sum_{s=1}^{r} \left| q_{r,s} \right| = \sum_{k=1}^{p} \left| q_{r,n_k}^{(1)} \right| \leqq 2K,$$

by (2), which is what is required.

Also, the columns of Q form null sequences, since they are either columns of $Q^{(1)}$, or they consist entirely of zeros.

Again $C=BQ=BQ^{(1)}D=BAC^{(1)}D=C^{(1)}D$, where B is the unique two-sided reciprocal of A. (The products are associative, since the matrices are all lower-semi.) Hence,

$$\sum_{j=1}^{i} | c_{i,j} | = \sum_{k=1}^{\lambda_i} | c_{i,n_k}^{(1)} |,$$

where λ_i is the number of 1's in the first i columns of D; thus

$$\sum_{j=1}^{i} | c_{i,j} | \geqq (\lambda_i - 1) | x_i |.$$

But $\lim_{i\to\infty} \lambda_i = \infty$, and $\limsup_{i\to\infty} |x_i| > 0$, since $\{x_i\}$ is divergent. Hence $\limsup_{i\to\infty} \sum_{j=1}^{i} |c_{i,j}| = \infty$, so that $\|C\| = \infty$.

Moreover, the columns of C are bounded, for they are either columns of $C^{(1)}$, or they consist entirely of zeros.

This completes the proof in the case where Q is required to be a lower-semi, but not necessarily normal, matrix.

To show that a normal matrix Q also exists, we note that $C^{(1)}$ and $Q^{(1)}$ are normal matrices, and that $I-D$ is a diagonal matrix of 0's and 1's whose nonzero elements occur only in those columns of D which consist entirely of zeros. Hence $\overline{C} \equiv C+I-D = C^{(1)}D+I-D$ is a normal matrix with bounded columns, and $\|\overline{C}\| = \infty$.

The corresponding Q, i.e., $\overline{Q}=A\overline{C}$, is normal (being the product of normal matrices), and, since $\overline{Q}=AC^{(1)}D+A(I-D)=Q^{(1)}D+A(I-D)$, we have $\|\overline{Q}\| \leq \|Q\| + \|A\| < \infty$; also the columns of $\overline{Q}$, which are all either columns of $Q^{(1)}$ or columns of A, form null sequences.

Hence the existence of a normal matrix Q with the required properties is proved.

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