QUESTIONS OF SIGNS IN POWER SERIES[1]

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This paper deals with the signs of the (real) coefficients of a power series $k(x)=\sum_{\nu=0}^{\infty} k_\nu x^\nu$, which is the quotient of the two power series $q(x)=\sum_{\nu=0}^{\infty} q_\nu x^\nu$ and $p(x)=\sum_{\nu=0}^{\infty} p_\nu x^\nu$. Questions of this type have been previously treated by Kaluza (1928) and Szegö (1926) in the special case of $q(x)=1$. Their significance for inclusion theorems for Nörlund means was shown by Hardy (1949), §4.5.

By consideration of a general power series $q(x)$ instead of $q(x)=1$ it is possible to include the known results[2] of this subject in a single theorem (§1.1). Moreover the method of proof used allows us to establish a second type of theorem using simpler conditions (§1.2). For this second type of theorem the introduction of a general $q(x)$ is essential for applications, because the results are still trivial for $q(x)=1$. There are two different principles to generalize. Thereby the second kind of theorem becomes also applicable in the case of $q(x)=1$ (§§2.1, 2.2, 2.4). The statements now obtained can even be used conversely for a second proof of the theorems first obtained (§2.3).

The results of this paper are, as in the book of Hardy, fundamentally for application to inclusion of Nörlund means and related questions. These applications will be discussed in a following paper *Some new inclusion theorems for Nörlund means*. Also some concrete examples for the general theorems on distribution of signs will be found there.

1. **Distribution of signs in $k(x)$.** In the following we begin always with the formal product

$$k(x)p(x) = q(x) \quad \text{or} \quad \sum k_\nu x^\nu \sum p_\mu x^\mu = \sum q_n x^n,$$

i.e. with the relation

$$\sum_{\nu=0}^{n} k_\nu p_{n-\nu} = q_n \qquad \text{for } n \geqq 0. \tag{1}$$

1.1. All the known results on the distribution of signs in $k(x) = \sum k_\nu x^\nu$ are included in

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[2] Another but similar question is that concerning the product of two power series instead of the quotient. Cf., for this, Davenport-Pólya (1949).

THEOREM 1. *If p_n and q_n satisfy the conditions*

$$(2) \qquad p_n > 0 \quad (n \geqq 0), \qquad \frac{p_{n+1}}{p_n} \nearrow \qquad \textit{for } 0 \leqq n \nearrow,$$[3]

and

$$(3) \qquad \frac{q_n}{p_n} \nearrow \qquad \textit{for } 0 \leqq n \nearrow,$$

then $k_n \geqq 0$ for $n \geqq 1$. (Hence it follows that either $\sum k_n = +\infty$ or $\sum k_n = \lim_{x \to 1-0} k(x)$, where both terms exist.)

To explain the contents of the theorem we consider first some specializations:

Because we placed no hypothesis on the signs of q_n, we may replace the terms q_n, k_n in Theorem 1 by $-q_n$, $-k_n$. In this way we obtain immediately

THEOREM 2. *If p_n and q_n satisfy* (2) *and*

$$(4) \qquad \frac{q_n}{p_n} \searrow \qquad \textit{for } 0 \leqq n \nearrow,$$[3]

then $k_n \leqq 0$ for $n \geqq 1$. (Hence it follows that either $\sum k_n = -\infty$ or $\sum k_n = \lim_{x \to 1-0} k(x)$, where both terms exist.)

Theorem 2 clearly contains the following result[4] of Kaluza (1928) and Szegö (1926).

THEOREM 3. *Let* (2) *hold and $q_0 = 1$, $q_n = 0$ $(n \geqq 1)$ (the latter means $k(x) = 1/p(x)$ in the formal sense).*

Then we have $k_0 = 1/p_0 > 0$, $k_n \leqq 0$ $(n \geqq 1)$. If moreover $p(x) = \sum p_\nu x^\nu$ is convergent for $|x| < 1$, then it follows that $\sum k_n = \lim_{x \to 1-0} 1/p(x) \geqq 0$, where both terms exist.

If in Theorem 1 we have always $q_n > 0$ (or only $q_0 > 0$), then $k_n \geqq 0$ for $n \geqq 0$ holds, because of $k_0 = q_0/p_0$. In this case the assertion of Theorem 1 can be taken from the proof of Hardy (1949) for his Theorem 23 (p. 69). By a short cut in the proof given there it is possible to deal with both cases $q_n < 0$ and $q_n > 0$ at the same time.

PROOF OF THEOREM 1. Let $n \geqq 1$ and if $n > 1$ assume moreover $k_1, \cdots, k_{n-1} \geqq 0$. Then we have to show that $k_n \geqq 0$. By (1) we have

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[3] The symbol "$\nearrow$" resp. "$\searrow$" means increasing resp. decreasing (equality allowed).

[4] Cf. Hardy (1949), p. 68, Theorem 22. An example for such a distribution of signs is to be found in Knopp (1926), p. 331.

$$(5)\qquad \sum_{\nu=0}^{n-1} k_\nu\left(\frac{p_{n-\nu}}{p_n}-\frac{p_{n-1-\nu}}{p_{n-1}}\right)+k_n\frac{p_0}{p_n}=\frac{q_n}{p_n}-\frac{q_{n-1}}{p_{n-1}}$$

and further by (2)

$$\frac{p_{n-\nu}}{p_n}-\frac{p_{n-1-\nu}}{p_{n-1}}\begin{cases}=0 & \text{for}\quad \nu=0,\\ \leqq 0 & \text{for}\quad 1\leqq\nu\leqq n-1.\end{cases}$$

Substituting in (5) gives by use of (3)

$$k_n\frac{p_0}{p_n}\geqq\frac{q_n}{p_n}-\frac{q_{n-1}}{p_{n-1}}\geqq 0,\qquad \text{q.e.d.}$$

1.2. An undesirable feature of the above theorem is the somewhat complicated condition (2). If $\sum p_\nu x^\nu$ is convergent for $|x|<1$ (as it will always be in later applications), then it follows from (2) that p_n is decreasing. The next theorems will use this weaker condition.

THEOREM 4. *If p_n and q_n satisfy the conditions*

$$(6)\qquad p_0>0,\qquad p_n\searrow \qquad \textit{for } 0\leqq n\nearrow$$

and

$$(7)\qquad \bar{\Delta}q_n\geqq\frac{q_0}{p_0}\bar{\Delta}p_n \qquad \textit{for } n\geqq 1,$$ [5]

then $k_n\geqq 0$ for $n\geqq 1$. (Hence it follows that either $\sum k_n=+\infty$ or $\sum k_n=\lim_{x\to 1-0}k(x)$, where both terms exist.)

As in Theorem 1 there is no supposition on the sign of q_n, and therefore we may replace again q_n, k_n by $-q_n$, $-k_n$. This gives immediately

THEOREM 5. *Let* (6) *and*

$$(8)\qquad \bar{\Delta}q_n\leqq\frac{q_0}{p_0}\bar{\Delta}p_n \qquad \textit{for } n\geqq 1.$$

Then we have $k_n\leqq 0$ for $n\geqq 1$. (Hence it follows that either $\sum k_n=-\infty$ or $\sum k_n=\lim_{x\to 1-0}k(x)$, where both terms exist.)

For the reciprocal power series $1/p(x)$ we obtain directly from Theorems 4 and 5 only two rather trivial results.

PROOF OF THEOREM 4. Let $n\geqq 1$ and for $n>1$ assume moreover $k_1,\cdots,k_{n-1}\geqq 0$. Then we have to show that $k_n\geqq 0$. By (1) we have

[5] We use the abbreviations $\bar{\Delta}a_\nu=a_\nu-a_{\nu-1}$ $(\nu\geqq 1)$, $\bar{\Delta}a_0=a_0$. Then the condition (7) is satisfied of itself for $n=0$.

$$k_0(p_n - p_{n-1}) + \sum_{\nu=1}^{n-1} k_\nu(p_{n-\nu} - p_{n-1-\nu}) + k_n p_0 = q_n - q_{n-1} \tag{9}$$

and further

$$k_0 = \frac{q_0}{p_0}, \qquad p_{n-\nu} - p_{n-1-\nu} \leqq 0 \qquad \text{for } 1 \leqq \nu \leqq n-1.$$

Substituting in (9) and applying (7) we obtain

$$k_n p_0 \geqq \bar{\Delta} q_n - \frac{q_0}{p_0} \bar{\Delta} p_n \geqq 0, \qquad \text{q.e.d.}$$

1.3. We now compare briefly the conditions of the Theorems 1 and 4. As already mentioned, (6) is in general weaker than (2). Of the other conditions—as a compensation so to speak—(7) is in general stronger than (3). More precisely we have

THEOREM 6. *If* $0 < p_n \searrow$ *for* $0 \leqq n \nearrow$, *then* (3) *follows from* (7).
If on the contrary $0 < p_n \nearrow$ *for* $0 \leqq n \nearrow$, *then conversely* (7) *follows from* (3).

PROOF. First we state at once the equivalence of the inequalities

$$\frac{q_n}{p_n} \geqq \frac{q_{n-1}}{p_{n-1}} \quad \text{and} \quad \bar{\Delta} q_n \geqq \frac{q_{n-1}}{p_{n-1}} \bar{\Delta} p_n$$

for $n \geqq 1$. Further by (3) or by (7) we have

$$\frac{q_{n-1}}{p_{n-1}} \geqq \frac{q_0}{p_0} \qquad \text{for } n \geqq 1.$$

Both together give the assertion with regard to the sign of $\bar{\Delta} p_n$.

2. Some extensions.

2.1. We have previously defined $k(x)$ by the formal quotient $q(x)/p(x)$. But in applying the Theorems 1 and 4 we can use as well the representation

$$k(x) = \frac{q(x)r(x)}{p(x)r(x)} = \frac{\bar{q}(x)}{\bar{p}(x)} \quad \text{with } r(x) = \sum r_\nu x^\nu, \tag{10}$$

and this gives

THEOREM 7. *If there is a sequence* r_n *such that the sequences*

$$\bar{q}_n = \sum_{\nu=0}^{n} q_\nu r_{n-\nu}, \qquad \bar{p}_n = \sum p_\nu r_{n-\nu}, \tag{11}$$

instead of $\bar{q}_n$, $\bar{p}_n$, *satisfy the conditions* (2), (3) *or* (6), (7), *then* $k_n \geqq 0$ *for* $n \geqq 1$.

The proof follows from the relation

$$\sum_{\nu=0}^{n} k_\nu \bar{p}_{n-\nu} = \bar{q}_n \qquad \text{for } n \geqq 0, \tag{12}$$

which we have by (1) and (11).

Corresponding to this we can also have extensions of Theorems 2, 3, and 5.

We obtain simple and at the same time general conditions by setting $r(x) = (1-x)^\alpha$, i.e.

$$r_n = (-1)^n \binom{\alpha}{n} = \binom{n-\alpha-1}{n}, \qquad \alpha \text{ any real number.} \tag{13}$$

In this case we shall write

$$\bar{q}_n = \bar{\Delta}^\alpha q_n, \qquad \bar{p}_n = \bar{\Delta}^\alpha p_n. \tag{14}$$

For example we formulate the part of Theorem 7, which corresponds to Theorem 4.

THEOREM 8. *If there is a real number* α *such that*

$$p_0 > 0, \qquad \bar{\Delta}^\alpha p_n \searrow \qquad \textit{for } 0 \leqq n \nearrow, \tag{15}$$

$$\bar{\Delta}^{\alpha+1} q_n \geqq \frac{q_0}{p_0} \bar{\Delta}^{\alpha+1} p_n \qquad \textit{for } n \geqq 1 \tag{16}$$

are satisfied, then $k_n \geqq 0$ *for* $n \geqq 1$.

2.2. In some cases it is not possible to have results directly concerning the distribution of signs in $k(x)$, but only in $k(x)r(x)$. By application of the Theorems 1 and 4 we find

THEOREM 9. *If there is a sequence* r_n *such that the sequences*

$$\bar{q}_n = \sum_{\nu=0}^{n} q_\nu r_{n-\nu} \textit{ and } p_n, \tag{17}$$

instead of q_n, p_n, *satisfy the conditions* (2), (3) *or* (6), (7), *then* $\bar{k}_n = \sum_{\nu=0}^{n} k_\nu r_{n-\nu} \geqq 0$ *for* $n \geqq 1$.

The proof follows at once from the relation

$$\sum_{\nu=0}^{n} k_\nu p_{n-\nu} = \bar{q}_n \qquad \text{for } n \geqq 0. \tag{18}$$

The situation is very similar to that of Theorem 7, and the examples shown there may be used in this case too. Thus instead of Theorem 8 we have

THEOREM 10. *If there is a real number* α *such that condition* (6),

$$p_0 > 0, \qquad p_n \searrow \qquad \text{for } 0 \leqq n \nearrow,$$

and

$$\bar{\Delta}^{\alpha+1} q_n \geqq \frac{q_0}{p_0} \bar{\Delta} p_n \qquad \text{for } n \geqq 1 \tag{19}$$

are satisfied, then $\bar{\Delta}^{\alpha} k_n \geqq 0$ *for* $n \geqq 1$.

COROLLARY. *If instead of* (19) *with* $\alpha = -1$ *we require the stronger condition*

$$q_n \geqq 0 \qquad \text{for } n \geqq 0, \tag{20}$$

then $\sum_{\nu=0}^{n} k_\nu \geqq 0$ *for* $n \geqq 0$.

For the proof put $r(x) = (1-x)^\alpha$ in Theorem 9 and use the conditions (6), (7).

The corollary refers to the especially simple case $\alpha = -1$. The condition (20) is satisfied e.g. for $q(x) = 1$, i.e. in the case of the reciprocal power series $k(x) = 1/p(x)$. This example shows what may be deduced from the condition (6) alone.

2.3. The last remark enables us to add a second proof of Theorem 4: First let formally $1/p(x) = \sum_{\nu=0}^{\infty} \gamma_\nu x^\nu$, i.e.

$$\sum_{\nu=0}^{n} p_\nu \gamma_{n-\nu} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \geqq 1. \end{cases} \tag{21}$$

Putting $\Gamma_n = \sum_{\nu=0}^{n} \gamma_\nu$ we have then for $n \geqq 1$

$$p_0 \Gamma_n + \sum_{\nu=1}^{n} \bar{\Delta} p_\nu \Gamma_{n-\nu} = 0, \tag{22}$$

which by induction gives $\Gamma_n \geqq 0$ for $n \geqq 0$ because of (6). On the other hand we have formally $k(x) = q(x)(1/p(x))$, hence for $n \geqq 1$

$$k_n = \sum_{\nu=0}^{n} q_\nu \gamma_{n-\nu} = q_0 \Gamma_n + \sum_{\nu=1}^{n} \bar{\Delta} q_\nu \Gamma_{n-\nu}. \tag{23}$$

Then from (7) and (22) follows

$$k_n \geqq \frac{q_0}{p_0} \cdot p_0 \Gamma_n + \sum_{\nu=1}^{n} \frac{q_0}{p_0} \cdot \bar{\Delta} p_\nu \Gamma_{n-\nu} = 0, \qquad \text{q.e.d.}$$

This method of proof is related to the method of Hardy's proof[6] for the special case of Theorem 1 previously mentioned.

2.4. As we have just now seen the considerations of signs in the reciprocal series $1/p(x)$ are fundamental for the theorems stated. Therefore it is interesting that we may get further theorems by combination of the methods of Theorems 7 and 9, which are applicable to the reciprocal series. For simplicity we restrict ourselves to a combination of the Theorems 8 and 10, which corresponds essentially to Theorem 4. There is of course a similar theorem resulting from Theorem 1.

THEOREM 11. *If there are two real numbers* α, β *such that*

$$p_0 > 0, \qquad \bar{\Delta}^{\alpha} p_n \searrow \qquad \text{for } 0 \leqq n \nearrow \tag{24}$$

and

$$\bar{\Delta}^{\alpha+\beta+1} q_n \geqq \frac{q_0}{p_0} \bar{\Delta}^{\alpha+1} p_n \qquad \text{for } n \geqq 1 \tag{25}$$

are satisfied, then $\bar{\Delta}^{\beta} k_n \geqq 0$ *for* $n \geqq 1$.

COROLLARY. *If instead of* (25) *with* $\beta = -\alpha - 1$ *we require the stronger condition*

$$q_n \geqq 0 \qquad \text{for } n \geqq 0, \tag{26}$$

then $\bar{\Delta}^{-\alpha-1} k_n \geqq 0$ *for* $n \geqq 0$.

The proof is trivial.

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[6] Cf. Hardy (1949), p. 69, Theorem 23.