A NON-EXCEPTIONAL ELEMENT OF WIENER SPACE

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Let C denote the space of functions $x(t)$, continuous on $0 \leqq t \leqq 1$ with $x(0)=0$. For each element of C let

$$\sigma_n(x) = \sum_{j=1}^{2^n} \left[x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right) \right]^2.$$

Cameron and Martin [1][1] have shown that for almost all elements of C, in the sense of Wiener measure, the limit

$$\int_0^1 | dx(t) |^2 = \lim_{n \to \infty} \sigma_n(x)$$

exists and has the value $1/2$.

The purpose of this note is to construct an explicit example of an element of C for which the above limit is equal to $1/2$. The example is of interest, since for most "ordinary" functions the value of the limit is zero. If $x(t)$ is of bounded variation on $0 \leqq t \leqq 1$, the limit is zero. If $x(t)$ satisfies for $0 \leqq t \leqq 1$ a uniform Lipschitz condition of order α, where α is greater than $1/2$, the limit is zero, whereas it is possible for such a function to be everywhere nondifferentiable [2].

The present construction is a modification of one given by the author [2]. In the notation of [2] we use $A=\alpha=1/2$. Specifically let $g(t, h)$ be periodic of period $2h$, equal to zero for even multiples of h, equal to one for odd multiples of h, and linear between. Let $x_0(t)$ be defined by

$$x_0(t) = \frac{1}{2} \sum_{m=1}^{\infty} 2^{-m/2} g(t, 2^{-m}).$$

$x_0(0)=0$, and $x_0(t)$ is continuous on $0 \leqq t \leqq 1$ by uniform convergence of the series. Thus $x_0(t) \in C$. We now show that

$$\int_0^1 | dx_0(t) |^2 = \frac{1}{2} \cdot$$

If the interval $0 \leqq t \leqq 1$ is divided into 2^n equal parts, the increments of $x_0(t)$ are the 2^n different values that can be obtained from all possible combinations of signs in the expression

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[1] The author has been informed that this result was found originally by P. Lévy.

$$\frac{1}{2} 2^{-n}(\pm 2^{n/2} \pm 2^{(n-1)/2} \pm \cdots \pm 2^{1/2}).$$

This follows from the fact that only the first n terms of the series for $x_0(t)$ will have increments different from zero, while the directions of the increments of the first n terms will occur in all possible combinations. Now when we compute the sum of the squares of these 2^n increments, the cross-product terms occur as often positive as negative, so that only the squares of the terms contribute, and the sum of the squares of the increments is

$$\begin{aligned} \sigma_n(x_0) &= \frac{1}{4} 2^{-2n} 2^n (2^n + 2^{n-1} + \cdots + 2) \\ &= \frac{1}{2} - \frac{1}{2^{n+1}} . \end{aligned}$$

Therefore

$$\lim_{n \to \infty} \sigma_n(x_0) = \int_0^1 | dx_0(t) |^2 = \frac{1}{2} .$$

BIBLIOGRAPHY

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