A TRANSFORMATION FOR S-FRACTIONS[1]

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1. Introduction. Stieltjes [1][2] has discussed the correspondence between certain formal power series,

$$Q(w) = \sum_{n=0}^{\infty} \mu_n w^{n+1}, \tag{1.1}$$

and S-fractions,

$$Q(w) \sim \frac{c_1 w}{1} + \frac{c_2 w}{1} + \cdots + \frac{c_m w}{1} + \cdots, \tag{1.2}$$

where $c_i \neq 0$, $i=1, 2, \cdots, m$, or $i=1, 2, \cdots$, according as the S-fraction terminates with the mth partial quotient or does not terminate. The correspondence (1.2) is characterized by the condition that $Q(w)$ and the S-fraction satisfy the formal power series identities

$$Q(w) - \frac{C_j(w)}{D_j(w)} = (-1)^j c_1 c_2 \cdots c_{j+1} w^{j+1}[1 + h_{j,1}w + \cdots]$$

$$(j = 0, 1, \cdots, m-1),$$

$$Q(w) - \frac{C_m(w)}{D_m(w)} = 0,$$

or

$$Q(w) - \frac{C_j(w)}{D_j(w)} = (-1)^j c_1 c_2 \cdots c_{j+1} w^{j+1}[1 + h_{j,1}w + \cdots]$$

$$(j = 0, 1, 2, \cdots)$$

according as the S-fraction terminates with the mth partial quotient or does not terminate, where $C_j(w)$ and $D_j(w)$ denote the jth numerator and denominator, respectively, of the S-fraction. The correspondence (1.2) relates to any S-fraction a unique formal power series (1.1) known as its corresponding power series. A sequence of numbers $\{\mu_n\}$ $(n=0, 1, 2, \cdots)$ such that the formal power series

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[1] Some of the material in this paper constitutes a part of a thesis submitted to Northwestern University and prepared under the direction of Professor Walter T. Scott.

[2] Numbers in brackets refer to references at the end of this paper.

(1.1) is the corresponding power series of an S-fraction will be called an S-sequence. If $\{\mu_n\}$ $(n=0, 1, 2, \cdots)$ is an S-sequence, the S-fraction expansion of the formal power series (1.1) determined by (1.2) is unique.

A sequence of real numbers $\{\mu_n\}$ $(n=0, 1, 2, \cdots)$ is called a Stieltjes moment sequence if there is a monotone nondecreasing function $g(t)$ on the interval $0 \leqq t < \infty$ such that

$$\mu_n = \int_0^\infty t^n dg(t) \qquad (n = 0, 1, 2, \cdots).$$

A sequence of real numbers $\{\mu_n\}$ $(n=0, 1, 2, \cdots)$, not all zero, is a Stieltjes moment sequence if and only if it is an S-sequence such that the coefficients c_i in (1.2) satisfy the conditions $c_1>0$ and $c_i<0$ for $i=2, 3, \cdots, m$ or $i=2, 3, \cdots$ according as the S-fraction terminates with the mth partial quotient or does not terminate.

A sequence of real numbers $\{\mu_n\}$ $(n=0, 1, 2, \cdots)$ will be called a monotone Hausdorff moment sequence if there is a monotone nondecreasing function $g(t)$ on the interval $0 \leqq t \leqq 1$ such that

$$\mu_n = \int_0^1 t^n dg(t) \qquad (n = 0, 1, 2, \cdots).$$

H. S. Wall [2] has shown that a sequence of real numbers $\{\mu_n\}$ $(n=0, 1, 2, \cdots)$, not all zero, is a monotone Hausdorff moment sequence if and only if the S-fraction expansion of the formal power series (1.1) is of the form

$$\frac{r_1 w}{1} - \frac{r_2 w}{1} - \frac{(1-r_2)r_3 w}{1} - \frac{(1-r_3)r_4 w}{1} - \cdots - \frac{(1-r_{m-1})r_m w}{1} - \cdots,$$

where

$$r_1 > 0; \qquad 0 < r_i < 1, \qquad i = 2, 3, \cdots, m-1; 0 < r_m \leqq 1$$

or

$$r_1 > 0; \qquad 0 < r_i < 1, \qquad i = 2, 3, \cdots,$$

according as the S-fraction terminates with the mth partial quotient or does not terminate.

In §2 we consider simultaneously the problems of embedding a given S-sequence $\{\mu_n\}$ in an S-sequence $\{\lambda_0, \lambda_1(=\mu_0), \lambda_2(=\mu_1), \cdots\}$

and of determining when a given S-sequence $\{\lambda_n\}$ is such that the sequence $\{\mu_0(=\lambda_1), \mu_1(=\lambda_2), \cdots\}$ is also an S-sequence. The conditions developed are in terms of certain parameters which enable us to obtain the S-fraction expansion of the formal power series $P(w)=\sum_{n=0}^{\infty}\lambda_n w^{n+1}$ from the S-fraction expansion of the formal power series $Q(W)=\sum_{n=0}^{\infty}\mu_n w^{n+1}$ in case $P(w)\equiv\lambda_0 w+wQ(w)$ and vice versa.

In §3 we apply the S-fraction transformation theorem of §2 for terminating S-fractions to the problem of extending backward a given Stieltjes moment sequence $\{\mu_n\}$ such that the S-fraction expansion of the formal power series (1.1) is terminating. In §4 we indicate how the results proved in §3 can be used to prove the basic theorem relative to the first backward extension of a given monotone Hausdorff moment sequence $\{\mu_n\}$ such that the S-fraction expansion of the formal power series (1.1) is terminating.[3] H. S. Wall, [3] and [4], has derived some results relative to the problem of extending backward a given Stieltjes moment sequence $\{\mu_n\}$ such that the S-fraction expansion of the formal power series (1.1) is nonterminating. The author has obtained and extended somewhat the principal results of [3] relative to the problem mentioned by using the S-fraction transformation theorem of §2 for nonterminating S-fractions.[4]

2. **The transformation theorems.** We shall need some information relating the number of partial quotients in the S-fraction expansions of the formal power series

$$Q(w) = \sum_{n=0}^{\infty} \mu_n w^{n+1}, \tag{2.1}$$

$$P(w) = \sum_{n=0}^{\infty} \lambda_n w^{n+1} \tag{2.2}$$

in case both (2.1) and (2.2) are the corresponding power series of S-fractions and their coefficients are related by

$$\lambda_n = \mu_{n-1} \qquad (n = 1, 2, \cdots). \tag{2.3}$$

Here, as well as subsequently in this paper, we shall for convenience at times allow the partial quotient $0\cdot w/1$ to be joined to a terminating S-fraction.

[3] The referee has indicated an alternate and somewhat more direct method for obtaining the results of §4. The method used in this paper for obtaining these results is given to show how they follow from results obtained in §3 and hence from the S-fraction transformation theorem of §2 for terminating S-fractions.

[4] Bull. Amer. Math. Soc. Abstract 60-4-516.

We shall use the following lemma. It is to be noted that the hypotheses of this and other lemmas in this section are at times slightly weaker than the assumption of an S-fraction expansion for a given formal power series (2.1) or (2.2). The principal merit of such weakened hypotheses is that the results of these lemmas are equally applicable to the discussion of both terminating and nonterminating S-fractions.

LEMMA 2.1. *If*

$$Q(w) \sim \frac{c_1 w}{1} + \frac{c_2 w}{1} + \cdots + \frac{c_{2k} w}{1} \tag{2.4}$$

$$(c_i \neq 0;\ i = 1, 2, \cdots, 2k-1),$$

and if a power series (2.2) *with coefficients satisfying* (2.3) *is related to a continued fraction*

$$\frac{a_1^{(0)} w}{1} + \frac{a_2^{(0)} w}{1} + \cdots + \frac{a_{2k+2}^{(0)} w}{1} + \cdots,$$

which is either terminating with at least $(2k+2)$ *partial quotients or nonterminating, by the formal power series identity*

$$P(w) - \frac{A_{2k+1}^{(0)}(w)}{B_{2k+1}^{(0)}(w)} = - a_1^{(0)} a_2^{(0)} \cdots a_{2k+2}^{(0)} w^{2k+2}\left[1 + g_{2k+1,1}^{(0)} w + \cdots\right],$$

where $A_{2k+1}^{(0)}(w)$ *and* $B_{2k+1}^{(0)}(w)$ *denote the* $(2k+1)$*th numerator and denominator, respectively, of this latter continued fraction, then*

$$a_1^{(0)} a_2^{(0)} \cdots a_{2k+2}^{(0)} = 0.$$

PROOF. Let $C_j(w)$ and $D_j(w)$ denote the jth numerator and denominator, respectively, of the continued fraction in (2.4). Since

$$P(w) \equiv \lambda_0 w + wQ(w),$$

we have that

$$\begin{aligned} w\left[\lambda_0 + \frac{C_{2k}(w)}{D_{2k}(w)}\right] - \frac{A_{2k+1}^{(0)}(w)}{B_{2k+1}^{(0)}(w)} &= \left[P(w) - \frac{A_{2k+1}^{(0)}(w)}{B_{2k+1}^{(0)}(w)}\right] - w\left[Q(w) - \frac{C_{2k}(w)}{D_{2k}(w)}\right] \\ &= - a_1^{(0)} a_2^{(0)} \cdots a_{2k+2}^{(0)} w^{2k+2}\left[1 + g_{2k+1,1}^{(0)} w + \cdots\right]. \end{aligned}$$

Since the constant term in both the polynomials $B^{(0)}_{2k+1}(w)$ and $D_{2k}(w)$ is 1, it follows then that

$$w[\lambda_0 D_{2k}(w) + C_{2k}(w)]B^{(0)}_{2k+1}(w) - A^{(0)}_{2k+1}(w)D_{2k}(w)$$
$$= - a_1^{(0)} a_2^{(0)} \cdots a_{2k+2}^{(0)} w^{2k+2} + [\ ]w^{2k+3} + \cdots.$$

Therefore

$$a_1^{(0)} a_2^{(0)} \cdots a_{2k+2}^{(0)} = 0$$

since the coefficient of w^{2k+2} on the left side of this last formal power series identity is 0.

In a similar manner we have

LEMMA 2.2. *If*

$$P(w) \sim \frac{a_1 w}{1} + \frac{a_2 w}{1} + \cdots + \frac{a_{2k+1} w}{1} \qquad (a_i \neq 0;\ i = 1, 2, \cdots, 2k), \tag{2.5}$$

and if the power series (2.1) *with coefficients satisfying* (2.3) *is related to a continued fraction*

$$\frac{c_1^{(0)} w}{1} + \frac{c_2^{(0)} w}{1} + \cdots + \frac{c_{2k+1}^{(0)} w}{1} + \cdots,$$

which is either terminating with at least $(2k+1)$ *partial quotients or nonterminating, by the formal power series identity*

$$Q(w) - \frac{C^{(0)}_{2k}(w)}{D^{(0)}_{2k}(w)} = c_1^{(0)} c_2^{(0)} \cdots c_{2k+1}^{(0)} w^{2k+1}[1 + h^{(0)}_{2k+1,1} w + \cdots],$$

where $C^{(0)}_{2k}(w)$ *and* $D^{(0)}_{2k}(w)$ *denote the* $2k$*th numerator and denominator, respectively, of this latter continued fraction, then*

$$c_1^{(0)} c_2^{(0)} \cdots c_{2k+1}^{(0)} = 0.$$

From Lemma 2.1 and Lemma 2.2 we have at once the following theorem for terminating S-fractions. In the corresponding theorem for nonterminating S-fractions, which is also an immediate consequence of Lemmas 2.1 and 2.2, the index k of the following theorem is replaced by ∞.

THEOREM 2.1. *If* (2.4) *holds, and if a power series* (2.2) *with coefficients satisfying* (2.3) *has an* S*-fraction expansion, then this* S*-fraction* *has either $2k$ or $(2k+1)$ partial quotients. Conversely, if (2.5) holds, and if the power series (2.1) with coefficients satisfying (2.3) has an S-fraction expansion, then this S-fraction has either $(2k-1)$ or $2k$ partial quotients.*

We now proceed with our development of the transformation theorems. We shall use the following lemma.

LEMMA 2.3. *Suppose the formal power series (2.1) is related to the terminating continued fraction*

$$(2.4)' \qquad \frac{c_1w}{1} + \frac{c_2w}{1} + \cdots + \frac{c_{2k}w}{1} \qquad (c_i \neq 0;\ i = 1, 2, \cdots, 2k-1)$$

by the formal power series identities

$$Q(w) - \frac{C_j(w)}{D_j(w)} = (-1)^j c_1c_2 \cdots c_{j+1}w^{j+1}[1 + h_{j,1}w + \cdots] \qquad (j = 0, 1, \cdots, 2k-1),$$

where $C_j(w)$ and $D_j(w)$ denote the jth numerator and denominator, respectively, of (2.4)'; moreover, suppose the formal power series (2.2) is related to the terminating continued fraction

$$(2.5)' \qquad \frac{a_1w}{1} + \frac{a_2w}{1} + \cdots + \frac{a_{2k+1}w}{1} \qquad (a_i \neq 0;\ i = 1, 2, \cdots, 2k)$$

by the formal power series identities

$$P(w) - \frac{A_j(w)}{B_j(w)} = (-1)^j a_1a_2 \cdots a_{j+1}w^{j+1}[1 + g_{j,1}w + \cdots] \qquad (j = 0, 1, \cdots, 2k),$$

where $A_j(w)$ and $B_j(w)$ denote the jth numerator and denominator, respectively, of (2.5)'. Then $\lambda_n = \mu_{n-1}$ $(n = 1, 2, \cdots, 2k)$ if and only if there is a set of parameters $(g_1, g_2, \cdots, g_{2k+1})$ satisfying

$$(2.6) \qquad g_i \neq 0 \quad (i = 1, 2, \cdots, 2k);\ g_{2j+1} \neq 1\ (j = 1, 2, \cdots, k-1)$$

and in terms of which (2.4)' and (2.5)' have the forms

$$(2.7) \qquad \begin{aligned} &\frac{g_1g_2w}{1} - \frac{g_2g_3w}{1} + \frac{(1-g_3)g_4w}{1} - \frac{g_4g_5w}{1} + \frac{(1-g_5)g_6w}{1} - \cdots \\ &- \frac{g_{2k-2}g_{2k-1}w}{1} + \frac{(1-g_{2k-1})g_{2k}w}{1} - \frac{g_{2k}g_{2k+1}w}{1} \end{aligned}$$

and

$$
(2.8)\quad \frac{g_1 w}{1} - \frac{g_2 w}{1} + \frac{g_2(1-g_3)w}{1} - \frac{g_3 g_4 w}{1} + \frac{g_4(1-g_5)w}{1} - \frac{g_5 g_6 w}{1} + \cdots + \frac{g_{2k-2}(1-g_{2k-1})w}{1} - \frac{g_{2k-1}g_{2k}w}{1} + \frac{g_{2k}(1-g_{2k+1})w}{1},
$$

respectively.

PROOF. From the formal power series identities

$$
\begin{aligned}
w\left[\lambda_0 + \frac{C_i(w)}{D_i(w)}\right] - \frac{A_{i+1}(w)}{B_{i+1}(w)}
&= \left[P(w) - \frac{A_{i+1}(w)}{B_{i+1}(w)}\right] - w\left[Q(w) - \frac{C_i(w)}{D_i(w)}\right] \\
&\quad - \left\{\sum_{n=1}^{\infty} \lambda_n w^{n+1} - w\sum_{n=0}^{\infty} \mu_n w^{n+1}\right\} \qquad (i = 0, 1, 2, \cdots, 2k)
\end{aligned}
$$

it follows that

$$
(2.9)\quad
\begin{aligned}
&w[\lambda_0 D_i(w) + C_i(w)]B_{i+1}(w) - A_{i+1}(w)D_i(w) \\
&= D_i(w)B_{i+1}(w)\left\{\left[P(w) - \frac{A_{i+1}(w)}{B_{i+1}(w)}\right] - w\left[Q(w) - \frac{C_i(w)}{D_i(w)}\right]\right. \\
&\quad \left. - \sum_{n=1}^{\infty} (\lambda_n - \mu_{n-1}) w^{n+1}\right\} \qquad (i = 0, 1, 2, \cdots, 2k).
\end{aligned}
$$

Since for each integer i ($i=0, 1, \cdots, 2k-1$) the constant term in both of the polynomials $D_i(w)$ and $B_{i+1}(w)$ is 1, we then have that

$$
(2.10)\quad
\begin{aligned}
&w[\lambda_0 D_i(w) + C_i(w)]B_{i+1}(w) - A_{i+1}(w)D_i(w) \\
&\quad = \{(-1)^{i+1}[(a_1 a_2 \cdots a_{i+2}) + (c_1 c_2 \cdots c_{i+1})]w^{i+2} \\
&\qquad + [\quad]w^{i+3} + \cdots\} - D_i(w)B_{i+1}(w)\sum_{n=1}^{\infty} (\lambda_n - \mu_{n-1}) w^{n+1} \\
&\qquad\qquad (i = 0, 1, \cdots, 2k-1).
\end{aligned}
$$

Suppose that $\lambda_n = \mu_{n-1}$ for $n = 1, 2, \cdots, 2k$. Then for each integer i ($i=0, 1, \cdots, 2k-1$) the coefficient of w^{i+2} on the right side of (2.10) is $(-1)^{i+1}$ $[(a_1 a_2 \cdots a_{i+2}) + (c_1 c_2 \cdots c_{i+1})]$. If $i=2j$ ($j=0, 1, \cdots, k-1$), the coefficient of w^{i+2} on the left side of (2.10) is 0, whereas if $i=2j+1$ ($j=0, 1, \cdots, k-1$), the coefficient of w^{i+2} on the left side of (2.10) is

$$
[(c_1 c_3 \cdots c_{2j+1})(a_2 a_4 \cdots a_{2j+2})].
$$

Therefore

$$
\begin{aligned}
&[(a_1a_2\cdots a_{2j+2})+(c_1c_2\cdots c_{2j+1})]=0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad (j=0,1,\cdots,k-1),\\
(2.11)\qquad &[(a_1a_2\cdots a_{2j+3})+(c_1c_2\cdots c_{2j+2})] \\
&\quad=(c_1c_3\cdots c_{2j+1})(a_2a_4\cdots a_{2j+2}) \qquad (j=0,1,\cdots,k-1).
\end{aligned}
$$

Using the parameters $(g_1, g_2, \cdots, g_{2k+1})$ given by

$$
(2.12)\qquad
\begin{aligned}
g_1 &= a_1,\\
g_{2j+2} &= \frac{c_1c_3\cdots c_{2j+1}}{a_1a_3\cdots a_{2j+1}} \qquad (j=0,1,\cdots,k-1),\\
g_{2j+3} &= \frac{c_2c_4\cdots c_{2j+2}}{a_2a_4\cdots a_{2j+2}} \qquad (j=0,1,\cdots,k-1),
\end{aligned}
$$

we have at once from (2.11) that

$$
\begin{aligned}
a_1 &= g_1, \qquad a_2 = -g_2,\\
a_{2j+1} &= g_{2j}(1-g_{2j+1}) \qquad (j=1,2,\cdots,k),\\
a_{2j+2} &= -g_{2j+1}g_{2j+2} \quad (j=1,2,\cdots,k-1);
\end{aligned}
$$

moreover, from these formulas and (2.11) it follows that

$$
\begin{aligned}
c_1 &= g_1g_2,\\
c_{2j} &= -g_{2j}g_{2j+1} \qquad (j=1,2,\cdots,k),\\
c_{2j+1} &= (1-g_{2j+1})g_{2j+2} \qquad (j=1,2,\cdots,k-1).
\end{aligned}
$$

Conversely, suppose that there is a set of parameters $(g_1, g_2, \cdots, g_{2k+1})$ satisfying (2.6) and in terms of which the continued fractions (2.4)′ and (2.5)′ have the forms (2.7) and (2.8), respectively. Then one can show by induction that these parameters are related to the coefficients $(c_1, c_2, \cdots, c_{2k})$ and $(a_1, a_2, \cdots, a_{2k+1})$ in (2.4)′ and (2.5)′, respectively, by (2.12). It can then be shown by induction that these coefficients are related by (2.11). Using the formal power series identities (2.10) and (2.11), we can show by induction that $\lambda_n=\mu_{n-1}$ for $n=1, 2, \cdots, 2k$.

This completes the proof of Lemma 2.3.

From Lemma 2.3 we have the following transformation theorem for terminating S-fractions. In the corresponding theorem for non-terminating S-fractions, which is an immediate consequence of Lemma 2.3, the conditions of the following theorem must hold for every index k; therefore, a statement of our result in this case is obtained by replacing k by ∞.

THEOREM 2.2. *Suppose that* (2.4) *and* (2.5) *hold. Then* (2.3) *holds if and only if there is a set of parameters* $(g_1, g_2, \cdots, g_{2k+1})$ *satisfying* (2.6) *and in terms of which the continued fractions in* (2.4) *and* (2.5) *have the forms* (2.7) *and* (2.8), *respectively.*

PROOF. The necessity of the above conditions for (2.3) to hold is an immediate consequence of Lemma 2.3.

Suppose, then, that there is a set of parameters $(g_1, g_2, \cdots, g_{2k+1})$ satisfying (2.6) and in terms of which the continued fractions in (2.4) and (2.5) have the forms (2.7) and (2.8), respectively. From Lemma 2.3 it follows that

$$\lambda_n = \mu_{n-1} \qquad (n = 1, 2, \cdots, 2k).$$

From (2.9) for $i = 2k$ we then have that

$$w[\lambda_0 D_{2k}(w) + C_{2k}(w)]B_{2k+1}(w) - A_{2k+1}(w)D_{2k}(w) = -D_{2k}(w)B_{2k+1}(w)\sum_{n=2k+1}^{\infty}(\lambda_n - \mu_{n-1})w^{n+1}.$$

The expression on the left side of this formal power series identity is a polynomial in w of degree less than $(2k+2)$. Therefore, using the fact that the constant term in both of the polynomials $D_{2k}(w)$ and $B_{2k+1}(w)$ is 1, we can show by induction that

$$\lambda_n = \mu_{n-1} \quad (n = 2k+1, 2k+2, \cdots).$$

This completes the proof of Theorem 2.2.

3. **An application to the backward extension of a Stieltjes moment sequence.** From Theorem 2.2 we have at once the following lemma.

LEMMA 3.1. *The correspondence*

$$(3.1) \qquad \sum_{n=0}^{\infty} \mu_n w^{n+1} \sim \frac{c_1 w}{1} + \frac{c_2 w}{1} + \cdots + \frac{c_{2k} w}{1}$$

$$(c_1 > 0;\ c_i < 0,\ i = 1, 2, \cdots, 2k-1;\ c_{2k} \leqq 0)$$

implies the correspondence

$$(3.2) \qquad \lambda_0 w + w\sum_{n=0}^{\infty} \mu_n w^{n+1} \sim \frac{a_1 w}{1} + \frac{a_2 w}{1} + \cdots + \frac{a_{2k+1} w}{1}$$

$$(a_1 > 0;\ a_i < 0,\ i = 1, 2, \cdots, 2k;\ a_{2k+1} \leqq 0)$$

if and only if (3.1) *has the form*

$$\sum_{n=0}^{\infty} \mu_n w^{n+1} \sim \frac{g_1 g_2 w}{1} - \frac{g_2 g_3 w}{1} + \frac{(1-g_3)g_4 w}{1} - \frac{g_4 g_5 w}{1} + \frac{(1-g_5)g_6 w}{1}$$

$$- \cdots - \frac{g_{2k-2}g_{2k-1}w}{1} + \frac{(1-g_{2k-1})g_{2k}w}{1} - \frac{g_{2k}g_{2k+1}w}{1}, \tag{3.3}$$

where

$$g_1 = \lambda_0, \qquad g_i > 0 \qquad (i = 1, 2, \cdots, 2k+1),$$

$$g_{2j+1} > 1 \qquad (j = 1, 2, \cdots, k-1), \tag{3.4}$$

$$g_{2k+1} \geqq 1;$$

in this case, (3.2) *has the form*

$$\lambda_0 w + w \sum_{n=0}^{\infty} \mu_n w^{n+1} \sim \frac{g_1 w}{1} - \frac{g_2 w}{1} + \frac{g_2(1-g_3)w}{1} - \frac{g_3 g_4 w}{1}$$

$$+ \frac{g_4(1-g_5)w}{1} - \frac{g_5 g_6 w}{1} + \cdots \tag{3.5}$$

$$+ \frac{g_{2k-2}(1-g_{2k-1})w}{1} - \frac{g_{2k-1}g_{2k}w}{1}$$

$$+ \frac{g_{2k}(1-g_{2k+1})w}{1}.$$

From Lemma 3.1 we have at once the following theorem.

THEOREM 3.1. *A Stieltjes moment sequence*

$$\{\mu_n\} \qquad (n = 0, 1, 2, \cdots) \tag{3.6}$$

such that

$$\sum_{n=0}^{\infty} \mu_n w^{n+1} \sim \frac{c_1 w}{1} + \frac{c_2 w}{1} + \cdots + \frac{c_{2k-1} w}{1} \tag{3.7}$$

$$(c_i \neq 0;\ i = 1, 2, \cdots, 2k-1)$$

cannot be extended backward once.

From Lemma 3.1, as well as Theorem 3.1, we have the following theorem.

THEOREM 3.2. *If* (3.6) *is a Stieltjes moment sequence such that*

$$\sum_{n=0}^{\infty} \mu_n w^{n+1} \sim \frac{c_1 w}{1} + \frac{c_2 w}{1} + \cdots + \frac{c_{2k} w}{1} \tag{3.8}$$

$$(c_i \neq 0;\ i = 1, 2, \cdots, 2k),$$

then

$$\{\lambda_0, \lambda_1(=\mu_0), \lambda_2(=\mu_1), \cdots\} \tag{3.9}$$

is a Stieltjes moment sequence if and only if

$$\lambda_0 \geqq -\left(\frac{c_1}{c_2} + \frac{c_1c_3}{c_2c_4} + \cdots + \frac{c_1c_3\cdots c_{2k-1}}{c_2c_4\cdots c_{2k}}\right); \tag{3.10}$$

moreover, in this case, (3.9) *can be extended backward once if and only if equality holds in* (3.10).

PROOF. Suppose that λ_0 is any real number satisfying (3.10). One can show by induction that the relations

$$g_1 = \lambda_0, \quad g_1g_2 = c_1, \ (-g_{2j}g_{2j+1}) = c_{2j} \qquad (j = 1, 2, \cdots, k)$$
$$(1 - g_{2j+1})g_{2j+2} = c_{2j+1} \qquad (j = 1, 2, \cdots, k-1)$$

can be solved for $(g_1, g_2, \cdots, g_{2k+1})$ and that these numbers are given by

$$g_1 = \lambda_0, \qquad g_2 = \frac{1}{c_1}\lambda_0, \qquad g_3 = -\frac{c_2}{c_1}\lambda_0,$$

$$g_{2j} = \left\{\left(\frac{c_2c_4\cdots c_{2j-2}}{c_1c_3\cdots c_{2j-1}}\right) \cdot\left[\lambda_0 + \left(\frac{c_1}{c_2} + \frac{c_1c_3}{c_2c_4} + \cdots + \frac{c_1c_3\cdots c_{2j-3}}{c_2c_4\cdots c_{2j-2}}\right)\right]\right\}^{-1} \quad (j = 2, 3, \cdots, k),$$

$$g_{2j+1} = -\left(\frac{c_2c_4\cdots c_{2j}}{c_1c_3\cdots c_{2j-1}}\right) \cdot\left[\lambda_0 + \left(\frac{c_1}{c_2} + \frac{c_1c_3}{c_2c_4} + \cdots + \frac{c_1c_3\cdots c_{2j-3}}{c_2c_4\cdots c_{2j-2}}\right)\right] \quad (j = 2, 3, \cdots, k). \tag{3.11}$$

Since the numbers

$$(g_1, g_2, \cdots, g_{2k+1})$$

given by (3.11) satisfy (3.4), it follows from Lemma 3.1 that (3.9) is a Stieltjes moment sequence. Moreover, since

$$g_{2k+1} = 1$$

if and only if equality holds in (3.10), we have from this result, as well as (3.5) and Theorem 3.1, that (3.9) can be extended backward once if and only if equality holds in (3.10).

The converse is immediate.

This completes the proof of Theorem 3.2.

4. **An application to the backward extension of a monotone Hausdorff moment sequence.** We shall indicate here a proof based on results of §3 for the following theorem; in the corresponding theorem for nonterminating S-fractions, the index k of the following theorem is in effect replaced by $+\infty$.

THEOREM 4.1. *If*

$$\{\mu_n\} \qquad (n = 0, 1, 2, \cdots) \tag{4.1}$$

is a monotone Hausdorff moment sequence such that

$$\begin{aligned}\sum_{n=0}^{\infty} \mu_n w^{n+1} &\sim \frac{r_1 w}{1} - \frac{r_2 w}{1} - \frac{(1-r_2)r_3 w}{1} - \frac{(1-r_3)r_4 w}{1} - \cdots \\ &\quad - \frac{(1-r_{2k-1})r_{2k} w}{1} \\ &= \frac{c_1 w}{1} + \frac{c_2 w}{1} + \cdots + \frac{c_{2k} w}{1} \\ &\qquad (c_i \neq 0;\ i = 1, 2, \cdots, 2k),\end{aligned} \tag{4.2}$$

then the sequence

$$\{\lambda_0, \lambda_1(=\mu_0), \lambda_2(=\mu_1), \cdots\} \tag{4.3}$$

is a monotone Hausdorff moment sequence if and only if

$$\lambda_0 \geqq -\left(\frac{c_1}{c_2} + \frac{c_1 c_3}{c_2 c_4} + \cdots + \frac{c_1 c_3 \cdots c_{2k-1}}{c_2 c_4 \cdots c_{2k}}\right). \tag{4.4}$$

Moreover, in case (4.4) *holds, then*

$$\begin{aligned}\sum_{n=0}^{\infty} \lambda_n w^{n+1} &\sim \frac{s_1 w}{1} - \frac{s_2 w}{1} - \frac{(1-s_2)s_3 w}{1} - \frac{(1-s_3)s_4 w}{1} - \cdots \\ &\quad - \frac{(1-s_{2k})s_{2k+1} w}{1} \\ &= \frac{a_1 w}{1} + \frac{a_2 w}{1} + \cdots + \frac{a_{2k+1} w}{1} \\ &\qquad (a_i \neq 0;\ i = 1, 2, \cdots, 2k),\end{aligned} \tag{4.5}$$

where

$$s_1 = \lambda_0, \qquad s_2 = r_2 \cdot \frac{1}{g_3},$$

$$(4.6) \quad s_{2j-1} = r_{2j-2} \cdot \frac{r_{2j-1} g_{2j+1}}{(1 - r_{2j-1}) r_{2j} + r_{2j-1} g_{2j+1}} \qquad (j = 2, 3, \cdots, k),$$

$$s_{2j} = \frac{(1 - r_{2j-1}) r_{2j} + r_{2j-1} g_{2j+1}}{g_{2j+1}} \qquad (j = 2, 3, \cdots, k)$$

in terms of the numbers $(g_3, g_5, \cdots, g_{2k+1})$ *given by*

$$g_3 = -\frac{c_2}{c_1} \lambda_0,$$

$$(4.7) \quad g_{2j+1} = -\left(\frac{c_2 c_4 \cdots c_{2j}}{c_1 c_3 \cdots c_{2j-1}}\right) \cdot \left[\lambda_0 + \left(\frac{c_1}{c_2} + \frac{c_1 c_3}{c_2 c_4} + \cdots + \frac{c_1 c_3 \cdots c_{2j-3}}{c_2 c_4 \cdots c_{2j-2}}\right)\right], \qquad (j = 2, 3, \cdots, k),$$

and where $a_{2k+1} = 0$ *in case equality holds in* (4.4) *or*

$$(4.6)' \qquad s_{2k+1} = r_{2k} \frac{(g_{2k+1} - 1)}{(1 - r_{2k}) + (g_{2k+1} - 1)}$$

in case strict inequality holds in (4.4).

It can be verified that if (4.4) holds then the numbers $(s_1, s_2, \cdots, s_{2k})$ given by (4.6) satisfy the conditions

$$(4.8) \qquad s_1 > 0; \; 0 < s_i < 1, \qquad i = 2, 3, \cdots, 2k - 1; \; 0 < s_{2k} \leqq 1,$$

and that if strict inequality holds in (4.4), then the numbers s_{2k} and s_{2k+1} given by (4.6) and (4.6)′, respectively, satisfy the conditions

$$(4.8)' \qquad 0 < s_{2k} < 1, \qquad 0 < s_{2k+1} \leqq 1.$$

Using the relationships

$$(4.9) \qquad g_{2j-1} - 1 = \frac{c_{2j-1}}{c_{2j}} g_{2j+1} = \frac{(1 - r_{2j-2}) r_{2j-1}}{(1 - r_{2j-1}) r_{2j}} g_{2j+1} \qquad (j = 2, 3, \cdots, k)$$

among the numbers $(g_3, g_5, \cdots, g_{2k+1})$ given by (4.7), it can be shown by induction that if (4.4) holds then the relations

$$s_1 = \lambda_0, \qquad s_2 = \frac{-c_2}{g_3},$$

$$(4.10) \qquad (1 - s_{2j-2})s_{2j-1} = \frac{c_{2j-2}(1 - g_{2j-1})}{g_{2j-1}} \qquad (j = 2, 3, \cdots, k),$$

$$(1 - s_{2j-1})s_{2j} = \frac{-c_{2j}g_{2j-1}}{g_{2j+1}} \qquad (j = 2, 3, \cdots, k)$$

can be solved for $(s_1, s_2, \cdots, s_{2k})$ and these numbers are given by (4.6), and that if strict inequality holds in (4.4) then the additional relation

$$(4.10)' \qquad (1 - s_{2k})s_{2k+1} = \frac{c_{2k}(1 - g_{2k+1})}{g_{2k+1}}$$

can be solved for s_{2k+1} and s_{2k+1} is given by (4.6)'. In view of Lemma 3.1 and Theorem 3.2, this completes the proof of Theorem 4.1.

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