TOPOLOGICAL INVARIANCE OF IDEALS IN MOBS[1]

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A *mob* is a Hausdorff space together with a continuous associative multiplication. *In all that follows S will be a compact mob.* A set $T \subset S$ is a *left ideal* if $T \neq \square$ and if $ST \subset T$. It is clear how to define right ideal and (two-sided) ideal. Numakura [7] has shown that S contains minimal ideals of all three sorts and these are closed sets. We let K be the *minimal ideal* of S. Improving some results of [11], we show among other things that, with additional assumptions on S, it is possible to give a *completely topological* definition of K. It will be seen also that if N is a sufficiently "large" sub*group* of S, then the cohomology structure of S is the same as that of N. This will be done by showing that $N = K$. From this it follows that N is a homomorphic retract of S. But N need not be a deformation retract of S, see [3].

The Alexander-Kolmogoroff cohomology group of the space X with coefficient group G will be denoted by $H^n(X, G)$, Spanier [8]. We sometimes write $H^n(X)$ for $H^n(X, G)$. It is possible to define a dimension function (Haskell Cohen [4]) by letting $cd(X, G) \leqq n$ if the natural homomorphism $H^n(X, G)$ into $H^n(A, G)$ is onto for each closed $A \subset X$. If X is compact Hausdorff then $cd(X$, integers) is the covering dimension, [1] and [5]. Cohen [4] showed that $cd(X, G)$ cannot exceed the covering dimension for a compact X. If $h \in H^n(X)$, then $h|A$ will denote the image of h in $H^n(A)$ under the natural homomorphism, $A \subset X$. A *continuum* is a compact connected Hausdorff space.

LEMMA. *Let A be a compact set in S and let Z be a continuum in S such that $cd(ZA, G) \leqq n$. Let $p, q \in Z$ and define $f: A \rightarrow qA$ by $f(x) = qx$. If $h \in H^n(qA, G)$ and if $h|(pA \cap qA) = 0$, then $f^*(h) = 0$. If also $q^2 = q$ and $qA \subset A$, then $h = 0$.*

PROOF. In the Mayer-Vietoris sequence [6, p. 43; 10] $H^n(qA \cup pA) \rightarrow H^n(qA) \times H^n(pA) \rightarrow H^n(qA \cap pA)$, the element $(h, 0)$ of the middle term goes into the zero of the last term, so that $(h, 0)$ is the image of an element $h_1 \in H^n(qA \cup pA)$. Since $cd(ZA) \leqq n$ and since $qA \cup pA$ is closed in ZA, we have $h_1 = h_2|(qA \cup pA)$ for some $h_2 \in H^n(ZA)$. Define $g: A \rightarrow pA$ by $g(x) = px$ and $g_0, f_0: A \rightarrow ZA$ by $g_0(x) = g(x)$, $f_0(x) = f(x)$. By [11, p. 47], we know that $f_0^* = g_0^*$. Now

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$$f_0^*(h_2) = f^*(h_2 \mid qA) = f^*(h)$$

and

$$g_0^*(h_2) = g^*(h_2 \mid pA) = g^*(0) = 0.$$

Hence $f^*(h)=0$. If $q^2=q$ and $qA\subset A$, then f is a retraction. Thus f^* is an isomorphism so that $f^*(h)=0$ gives $h=0$.

If X is compact Hausdorff and if $h\in H^p(X, G)$ is not zero, then there is a closed set $F\subset X$ such that $h|F\neq 0$, but $h|F_0=0$ for any closed proper subset F_0 of F. We term F a *floor* for h, see [**9**].

Remark. If S is *compact*, if A is closed in S, and if $t_0\in S$, then $A\subset t_0A$ implies $A=t_0A$ and also $A=eA$ for some $e\in S$, with $e^2=e$ [**12**, p. 24].

Theorem. *Let S be a compact connected mob with $cd(S, G)\leqq n$ and let N be a closed set in S with $H^n(N, G)\neq 0$ and with $N\subset t_1N$ for some $t_1\in S$. Then K, the minimal ideal of S, is also a minimal right ideal and K contains every floor for every nonzero $h\in H^n(N, G)$ and each such floor is a left ideal of S. If also $N\subset Nt_2$ for some $t_2\in S$, then K is a group and is the unique floor for each nonzero h in $H^n(N, G)$.*

Proof. Let h be a nonzero element of $H^n(N)$ and let A be a floor for h. Since $N\subset t_1N$, we have $N=eN$ for some $e\in S$ with $e^2=e$. Hence $A=eA$ because $A\subset N=eN$. Let $h_0=h|A$ and let $t\in S$. Now A is a floor for h_0, so that if A is not contained in tA then $A\cap tA$ is a proper subset of A, and thus $h_0|(eA\cap tA)=0$, recalling that $A=eA$. By the lemma, $h_0=0$ contrary to the fact that A is a floor for h. Thus $A\subset tA$ and hence $A=tA$. Take any $f\in K$ with $f^2=f$, see [**2**, p. 525]. Then $A=fA\subset fS\subset K$ and fS is a minimal right ideal. Now all minimal right ideals are obtainable as fS for some $f\in K$ with $f^2=f$ and because K is the union of all minimal right ideals, we see that $K=fS$ for any such f and so K is a minimal right ideal, see [**2**]. It is clear that A is a left ideal. If also $N\subset Nt_2$, then by left-right duality we have $A=K$.

Hence we see that K is a group and if e is the unit of K, then $xe=ex$ for each $x\in S$ and $x\to xe$ is a retracting homomorphism, see [**3**]. It also follows that (taking $S=N$) K can be defined as the unique floor for any nonzero $h\in H^n(S, G)$, so that K is a topological invariant of S in the following sense: Let S be a clan (=compact connected mob with (two-sided) unit), let $cd(S, G)\leqq n$, and let $H^n(S, G)\neq 0$. If T is a mob with unit and if f is a homeomorphism of S onto T, then f takes the minimal ideal of S onto the minimal ideal of T. The hypothesis $H^n(S, G)\neq 0$ is essential to this result.

Corollary. *Let S be a clan and for some coefficient group G_0, let*

$cd(S, G_0) \leqq n$ and let N be a closed subgroup of S with $H^n(N, G_0) \neq 0$. Then $K = N$ and hence N is a homomorphic retract of S and $H^p(N, G)$ is naturally isomorphic with $H^p(S, G)$ for any $p \geqq 0$ and any coefficient group G.

PROOF. By the theorem we know that $K \subset N$ so that $K = N$ because N is a group. That $H^p(S, G) \approx H^p(K, G)$ is known, [11, p. 48].

BIBLIOGRAPHY

1. P. Alexandroff, *On the dimension of normal spaces*, Proc. Roy. Soc. London Ser. A vol. 189 (1947) pp. 11–39.

2. A. H. Clifford, *Semigroups containing minimal ideals*, Amer. J. Math. vol. 70 (1948) pp. 521–526.

3. A. H. Clifford and D. D. Miller, *Semigroups having zeroid elements*, Amer. J. Math. vol. 70 (1948) pp. 117–125.

4. H. Cohen, *A cohomological definition of dimension for locally compact Hausdorff spaces*, Duke Math. J. vol. 21 (1954) pp. 209–224.

5. H. Dowker, *Mapping theorems for non-compact spaces*, Amer. J. Math. vol. 69 (1947) pp. 200–242.

6. S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton, 1952.

7. K. Numakura, *On bicompact semigroups*, Math. J. of Okayama Univ. vol. 1 (1952) pp. 99–108.

8. E. Spanier, *Cohomology theory for general spaces*, Ann. of Math. vol. 49 (1948) pp. 407–427.

9. A. D. Wallace, *Outline for algebraic topology*, Tulane University, 1949–1952.

10. ———, *The map excision theorem*, Duke Math. J. vol. 19 (1952) pp. 177–182.

11. ———, *Cohomology, dimension and mobs*, Summa Brasil. Math. vol. 3 (1953) pp. 43–54.

12. ———, *Inverses in Euclidean mobs*, Math. J. of Okayama Univ. vol. 3 (1953) pp. 23–28.

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