HAAR MEASURE AND THE SEMIGROUP OF MEASURES ON A COMPACT GROUP

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1. **Introduction.** Crucial results in the theory of a compact topological semigroup S state that S *must possess idempotents*, and that *if S contains an identity and is not a group then at least one additional idempotent occurs* [**6**; **7**; **8**]. These facts were pointed out to the author by R. J. Koch, whom it is here a pleasure to thank for many stimulating and instructive conversations.

An interesting example is furnished by the set S of (normalized nonnegative regular) measures on a compact group G; S becomes a compact Hausdorff semigroup with identity under natural definitions of multiplication and topology, and S is not a group if G has more than one element. Then S has additional idempotents; what are they? If the existence of Haar measures is assumed it is easy to show that the Haar measure associated with any compact subgroup of G determines an idempotent. The converse is more interesting, and is our main concern here: *the existence of idempotents in S implies the existence of Haar measure on G.* (At this writing we are unable to apply the method to general locally compact groups.) As a secondary result we show that *G is determined by S.*

2. **Definitions and preliminary results.** Let G be a compact group and let S be the set of countably additive nonnegative regular set functions μ on the Borel sets of G with $\mu(G)=1$. Let $C(G)$ be the Banach space of real continuous functions on G, and recall the 1-1 correspondence between measures $\mu\in S$ and continuous linear functionals ϕ on $C(G)$, with the properties $f(x)\geqq 0$ implies $\phi(f)\geqq 0$ and $\phi(1)=1$, that is given by the equation $\phi(f)=\int f(x)d\mu(x)$. For later reference we note the formula (cf. [**3**, Theorem 56. E])

$$\mu(U) = \sup\left\{\int f(x)d\mu(x) \,\middle|\, f\in C(G),\ 0\leqq f(x)\leqq 1, f(x)=0 \text{ for } x\notin U\right\} \tag{1}$$

valid for all open sets U in G.

We give S the weak* topology for functionals, so that $\mu_\alpha\to\mu$ means

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$\int f(x)d\mu_\alpha(x) \to \int f(x)d\mu(x)$, $f \in C(G)$. S is compact in this topology, by virtue of the weak* compactness of the unit sphere in the conjugate space of $C(G)$ and the fact that the restrictions on ϕ's imposed above determine a closed subset of the unit sphere.

As in [2] we define multiplication of μ, $\nu \in S$ by

$$\int f(x)d(\mu\nu)(x) = \int\int f(yz)d\mu(y)d\nu(z), \qquad f \in C(G); \tag{2}$$

it is easy to see that the multiplication is associative and continuous on S. Thus *S is a compact topological semigroup.*

Now for any element $x \in G$, we define the element $x' \in S$ as the point mass at x, i.e. $x'(E) = 1$ if $x \in E$, 0 if $x \notin E$. The corresponding functional sends the function f into the number $f(x)$, and the group element xy goes over into the measure $(xy)' = x'y'$. Therefore the mapping $x \to x'$ of G into S is a homeomorphic isomorphism, so that henceforth we may regard G as embedded in S and omit primes. Clearly the identity e of G is also the identity for S; in fact, making use of (2) it is easy to see that for $x \in G$

$$(\mu x)(E) = \mu(Ex^{-1}), \qquad (x\mu)(E) = \mu(x^{-1}E), \qquad \text{all } E.$$

The semigroup S has two special structural features not shared by all compact semigroups with identity, which are: the possibility of forming convex linear combinations of elements of S, and the presence of a natural antiautomorphic involution in S; it is doubtful whether we have made maximal use of these properties, but at any rate they turn out to be very useful.

More explicitly, for μ, $\nu \in S$ and t real, $0 \leqq t \leqq 1$, $t\mu + (1-t)\nu$ is again in S. This remark enables us to exhibit elements of S which do not possess inverses if G has more than one element, so that *S is not a group.* In fact, for $x, y \in G$ the measure $(1/2)x + (1/2)y$ has no inverse; this is a consequence of the following lemma whose proof we defer to the next section.

LEMMA 1. *A necessary and sufficient condition that $\mu \in S$ have an inverse is that μ is a point mass, $\mu \in G$.*

From this it follows easily that *S determines G*, for in order to reconstruct G from S we have only to pick out the elements with inverses.

The involution $\mu \to \mu^*$ is defined by $\mu^*(E) = \mu(E^{-1})$; the assertion $\mu^{**} = \mu$ is trivial, and the fact that $(\mu\nu)^* = \nu^*\mu^*$ follows easily from $\int f(x)d\mu^*(x) = \int f(x^{-1})d\mu(x)$ and equation (2). A deeper fact about the involution is contained in

LEMMA 2. *If μ is idempotent, then $\mu^* = \mu$.*

Again the proof is postponed to the next section.

3. **The idempotents of S.** Let H be a compact subgroup of G and suppose that H has Haar measure μ. We extend the definition of μ to all the Borel sets of G by putting $\mu(E) = \mu(E \cap H)$, and show that μ is idempotent. This follows from the more general statement that if $\nu \in S$ is carried on H (this means that $\nu(H) = 1$), then μ annihilates ν, $\mu\nu = \nu\mu = \mu$. In fact, to show that $\mu\nu = \mu$ we have $\int f(x) d(\mu\nu)(x) = \int\int f(yz) d\mu(y) d\nu(z) = \int_H \int_H f(yz) d\mu(y) d\nu(z) = \int_H \int_H f(y) d\mu(y) d\nu(z) = \{\int_H f(y) d\mu(y)\} \nu(H) = \int f(x) d\mu(x)$, $f \in C(G)$, and similarly $\nu\mu = \mu$.

We remark two important special cases: if $H = \{e\}$ then $\mu = e$, and if $H = G$ then $\mu =$ Haar measure of $G =$ annihilator of S. Of course here we are assuming that G has Haar measure.

Conversely, without assuming the existence of Haar measure, we are going to prove

THEOREM 1. *Let μ be an idempotent in S. There exists a unique compact subgroup H in G such that μ is the Haar measure of H, extended as above to all of G.*

Before proving the theorem we need some lemmas, as follows.

LEMMA 3. *For any $\mu \in S$ there is a unique closed set A, the carrier of μ, such that $\mu(A) = 1$ and $\mu(U) > 0$ for any nonempty relatively open subset U of A.*

PROOF. The uniqueness of A is easy, for if A' were a second closed set with the specified properties we could choose the notation so that $U = A' - A$ would be nonempty and write

$$1 = \mu(G) \geqq \mu(A) + \mu(A' - A) = 1 + \mu(A' - A) > 1,$$

a contradiction.

For the proof of existence let $\mathcal{J}$ be the family of closed sets F with $\mu(F) = 1$. $G \in \mathcal{J}$; hence $\mathcal{J}$ is not empty. For $F_1, F_2, \cdots, F_n \in \mathcal{J}$ we find that $\bigcap_i F_i$ again belongs to $\mathcal{J}$, since the complements of the F_i are μ-nullsets. Thus $\mathcal{J}$ has the finite intersection property; let A be the closed nonempty set of points common to all the F's. Let U be an open set containing A. Then U contains some finite intersection of F's. Therefore $\mu(U) = 1$. Then the regularity of μ shows that $\mu(A) = 1$. No closed proper subset of A belongs to $\mathcal{J}$, i.e. has μ-measure equal to 1, and this observation completes the proof of the lemma.

LEMMA 4. *Let A and B be the carriers of μ and ν. Then $AB = \{xy \mid x \in A, y \in B\}$ is the carrier of $\mu\nu$.*

We shall first show that the desired relation follows from

$$(3)\qquad (\mu\nu)(VW) \geqq \mu(V)\nu(W), \qquad V \text{ and } W \text{ open in } G.$$

Let U be an open set containing AB; by the continuity of multiplication we may find open V and W containing A and B such that VW is contained in U. Then

$$(\mu\nu)(U) \geqq (\mu\nu)(VW) \geqq \mu(V)\nu(W) \geqq \mu(A)\nu(B) = 1;$$

hence by the regularity of $\mu\nu$, $(\mu\nu)(AB)=1$. Similarly, if U is open in AB it follows easily that $(\mu\nu)(U)>0$. Therefore AB, which is closed by a known theorem, has the required properties.

It remains to establish (3), which we shall do by invoking (1). Let g and h be continuous maps of G to $[0, 1]$ vanishing outside of V and W respectively. Define a real function f on G by $f(z)=\max_x g(x)h(x^{-1}z)$. Clearly f is continuous, f sends G to $[0, 1]$, and f vanishes outside of VW. Moreover, $f(xy)\geqq g(x)h(y)$. Thus $\int f(z)d(\mu\nu)(z) =\int\int f(xy)d\mu(x)d\nu(y)\geqq \int g(x)d\mu(x)\int h(y)d\nu(y)$; taking suprema over g and h and applying (1) then yields (3).

Proof of Lemma 1. If $\mu\nu=e$ and A, B are the carriers of μ, ν, then $AB=\{e\}$. Hence A and B are one-point subsets of G, as had to be shown.

Lemma 5. *If μ is idempotent and H its carrier, then H is a compact subgroup of G.*

Proof. By Lemma 4, $H^2=H$. Therefore by a known theorem [**1**; **6**] H is a compact subgroup.

Proof of Theorem 1. Let H be the carrier of μ. By Lemma 3 every open set of H has positive μ-measure and, by Lemma 5, H is a compact subgroup of G. To show that μ is translation-invariant on H we must show that for any $f\in C(H)$ the expression

$$g(y) = \int f(xy)d\mu(x)$$

is constant for $y\in H$. The function g is continuous and attains its maximum at a point which, after replacing f by a suitable translate, we may take to be $y=e$. Then from (2) we have

$$g(e) = \int f(x)d\mu(x) = \int\int f(yz)d\mu(y)d\mu(z) = \int g(z)d\mu(z)$$
$$\leqq \int g(e)d\mu(z) = g(e).$$

Therefore g is identically $g(e)$, and since the uniqueness of H is trivial Theorem 1 is proved.

Proof of Lemma 2. This is an easy consequence of Theorem 1 and the known fact applied to H that on a compact group Haar measure is invariant under inverse.

4. **The existence of Haar measure.** Now we want to apply the foregoing considerations to give a new proof of the existence of Haar measure for compact groups. Let $G \neq \{e\}$; then, as we have seen, S is a compact semigroup with identity, which is not a group. There is then another idempotent in S, and by Theorem 1 some nontrivial subgroup H of G has Haar measure. The idea of the proof is to show first that there is a largest such subgroup H, which is necessarily normal in G; then if H is not already G we shall show that we have a contradiction.

We begin by partially ordering the compact subgroups H of G which have Haar measure under inclusion. The results of the previous section show that this is equivalent to partially-ordering the idempotents of S by annihilation: $\mu \leqq \nu$ means $\mu\nu = \nu\mu = \nu$.

Lemma 6. *The set of idempotents is a directed set in the above ordering.*

Proof. Let μ, ν be distinct idempotents of S. Form $\sigma = \mu\nu$ and let T be the closure in S of the set of powers of σ. It is clear that T is a compact subsemigroup of S. Since $\mu\sigma^n = \sigma^n\nu = \sigma^n$ we have $\mu\tau = \tau\nu = \tau$ for $\tau \in T$. (If G (and therefore S) is abelian, then trivially $T = \{\sigma\} = \{\mu\nu\}$.) By the fundamental theorem there is an idempotent τ in T. Applying the involution and Lemma 2 we find that τ annihilates both μ and ν. That is, $\tau \geqq \mu$, $\tau \geqq \nu$. (Recent results of Koch [4] show that τ is unique; from this it follows easily that the carrier of τ is the closed subgroup generated by the carriers of μ and ν.)

Lemma 7. *There is a greatest idempotent in S.*

Proof. The set of idempotents is defined by the relation $\mu^2 = \mu$ and therefore is compact. For each μ let $Q(\mu) = \{\nu \mid \nu \geqq \mu\}$. These sets are closed, since they are defined by relations $\nu^2 = \nu = \nu\mu = \mu\nu$. By Lemma 6 the family of $Q(\mu)$'s has the finite intersection property, and hence there is a point common to all of them. Another application of Lemma 6 shows that there is precisely one such point, and this is the desired greatest idempotent.

Lemma 8. *Let H be the carrier of the greatest idempotent μ. Then H is a nontrivial compact normal subgroup of G.*

Proof. H is a compact subgroup of Theorem 1: $H \neq \{e\}$ since e is certainly not the greatest idempotent. Finally we show that H is normal. Let $x \in G$. Then the measure $x^{-1}\mu x$ is idempotent, and its carrier is $x^{-1}Hx$, by Lemma 4. We must have $x^{-1}\mu x = \mu$ with the consequent $x^{-1}Hx = H$, since otherwise $\mu > x^{-1}\mu x$, from which it would easily follow that $x\mu x^{-1} > \mu$, a contradiction.

THEOREM 2. *G has Haar measure.*

PROOF. If H of Lemma 8 is G we are finished. But if $H \neq G$ we can construct an idempotent greater than μ in the following way. As shown in the proof of Lemma 8, μ commutes with each element of G. Then also μ commutes with each $\nu \in S$, for $\int\int f(xy)d\mu(x)d\nu(y) = \int\int f(yxy^{-1}y)d\mu(yxy^{-1})d\nu(y) = \int\int f(yx)d\mu(x)d\nu(y) = \int\int f(xy)d\nu(x)d\mu(y)$. Let $S' = S\mu$, a compact semigroup with identity μ. S' is not a group, for if $x \notin H$ the element $(1/2)x\mu + (1/2)\mu$ can have no inverse in $S' = S\mu$, by an argument with carriers similar to that used in proving Lemma 1. Then S' contains an idempotent $\nu = \nu\mu \neq \mu$, and $\mu\nu\mu = \nu\mu\mu = \nu\mu > \mu$. This contradiction completes the proof.

It may be worthwhile to sketch the mechanism underlying this proof. One might attempt to prove Theorem 2 as follows. If $H \neq G$, form the compact group $G' = G/H$, and find therein a nontrivial subgroup K' having Haar measure; by a modification of a device used in [**3**; **5**] one can then construct Haar measure on K, the antecedent of K' in the natural map of G upon G'; K properly contains H, a contradiction. The precise connection between these remarks and the given proof is found in

THEOREM 3. *$S' = S\mu$ may be identified with the semigroup $S(G')$ of measures on $G' = G/H$. (This does not depend on maximality of H, but only on its normality in G.)*

PROOF. Let $\nu' \in S(G')$ and $f \in C(G)$. Form $F(x) = \int f(xy)d\mu(y) = \int\int f(xyz)d\mu(y)d\mu(z)$. F is continuous and is constant on cosets x' of H and therefore defines an element of $C(G')$. Form $\int F(x)d\nu'(x')$, where x is any element of x'. This defines a functional of the appropriate kind on $C(G)$, and so may be written in the form $\int f(x)d\nu(x) = \int\int f(xz)d\nu(x)d\mu(z)$ with $\nu \in S$. Hence $\nu = \nu\mu \in S\mu = S'$. It is routine to verify that the mapping $\nu' \to \nu\mu$ is a continuous (and therefore homeomorphic) isomorphism of $S(G')$ onto S'; we suppress further details.

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ON THE SPECTRA OF COMMUTATORS[1]

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1. The following theorems will be proved:

(*) *Let A and B denote bounded operators in a Hilbert space and suppose that the commutator*

$$C = AB - BA \tag{1}$$

satisfies the commutation relation

$$AC = CA. \tag{2}$$

Then there exists a sequence of bounded operators $C_1, C_2, \cdots$ such that

$$\|C - C_n\| \to 0 \quad \textit{and} \quad \operatorname{sp}(C_n) \to 0 \textit{ as } n \to \infty. \tag{3}$$

(**) *If, in addition to the assumption* (2) *of* (*), *the relation*

$$BC = CB \tag{4}$$

also holds, then the assertion (3) *can be improved to*

$$\operatorname{sp}(C) \textit{ consists of } 0 \textit{ alone.} \tag{5}$$

For the terminology see, e.g., [3, pp. 9–10]. It is understood that sp (D) denotes the set of points in the spectrum of an operator D. Thus, the relation sp $(C_n)\to 0$ means that for any $\epsilon > 0$ there is a number N_ϵ such that the spectrum of C_n is contained in the disk $|\lambda| < \epsilon$ whenever $n > N_\epsilon$.

As a consequence of (**) one obtains the

COROLLARY. *If A is a bounded operator in a Hilbert space which com-*

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