# FINITELY GENERATED EXTENSIONS OF DIFFERENCE FIELDS

RICHARD M. COHN

Let $\mathcal{J}$, $\mathcal{H}$, $\mathcal{K}$ be difference fields such that $\mathcal{J}\subseteq\mathcal{H}\subseteq\mathcal{K}$. We shall prove that *if $\mathcal{K}$ is a finitely generated extension of $\mathcal{J}$, $\mathcal{K}=\mathcal{J}\langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$,*[1] *then $\mathcal{H}$ is also a finitely generated extension of $\mathcal{J}$.*

We introduce a new notation for the $\alpha_i$. Let $\beta_1, \cdots, \beta_q$ denote a subset of the $\alpha_i$ annulling no nonzero difference polynomial with coefficients in $\mathcal{H}$ and such that each $\alpha_i$ annuls some nonzero difference polynomial with coefficients in $\mathcal{H}\langle\beta_1, \cdots, \beta_q\rangle$. We denote the $\alpha_i$ not included among the $\beta_i$ by $\gamma_1, \cdots, \gamma_p$, $p=n-q$.

Let $\Lambda$ be the reflexive prime difference ideal in $\mathcal{H}\{u_1, \cdots, u_q; y_1, \cdots, y_p\}$ with the generic zero $u_i=\beta_i$, $i=1, \cdots, q$; $y_j=\gamma_j$, $j=1, \cdots, p$. We denote a characteristic set of $\Lambda$ by

$$(1) \qquad A_{10}, \cdots, A_{1k_1};\quad A_{20}, \cdots, A_{2k_2}; \cdots;\quad A_{p0}, \cdots, A_{pk_p},$$

where $A_{i0}$ introduces[2] $y_i$. Let $\mathcal{G}$ be the difference field formed by adjoining the coefficients[3] of the $A_{ij}$ to $\mathcal{J}$. Evidently $\mathcal{G}\subseteq\mathcal{H}$. The result stated above will follow when we show that $\mathcal{G}=\mathcal{H}$.

We shall describe what we mean by the characteristic sequences $B_{ij}$, $i=1, \cdots, p$; $j=0, 1, \cdots$, of $\Lambda$ formed from (1). This concept has been previously defined only in special cases.

Let $t_i$ denote the order of $A_{i0}$ in $y_i$. We let $B_{10}=A_{10}$. Suppose $B_{10}, \cdots, B_{1,k-1}$ have been defined. Then, if there is an $A_{1j}$ of order $t_1+k$ in $y_1$, we let $B_{1k}$ be that $A_{1j}$. Otherwise $B_{1k}$ is defined as the remainder[4] of the transform of $B_{1,k-1}$ with respect to the chain $B_{10}, \cdots, B_{1,k-1}$. It is easy to see that, for any $r$, $B_{1r}$ is of order $t_1+r$ in $y_1$ and, unless it is equal to some $A_{1j}$, of the same degree in the $(t_1+r)$th transform of $y_1$ as is $B_{1,r-1}$ in the $(t_1-1+r)$th transform of $y_1$.

Let $B_{20}=A_{20}$. Suppose $B_{20}, \cdots, B_{2,k-1}$ have been defined. Then if

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[1] The brackets $\langle\ \rangle$ denote field adjunction of the enclosed elements and their transforms so as to form a difference field. Similarly, brackets $\{\ \}$ denote ring adjunction of the enclosed elements and their transforms. Field and ring adjunctions in the usual sense are denoted by brackets ( ) and [ ] respectively. For other terms used see [1] (where the term "basic set" corresponds to our "characteristic set") and [3].

[2] If $p=0$, $\Lambda$ is the ideal consisting only of 0, and no $A_{ij}$ are defined.

[3] If $p=0$, we define $\mathcal{G}$ to be $\mathcal{J}$.

[4] Throughout this discussion we form the remainder treating the $B_{ij}$ not as difference polynomials but as polynomials as in Chapter IV of [2]. The $y_{ij}$, $j\geqq t_i$, are ordered lexicographically. The remaining $y_{ij}$ and the $u_{ij}$ precede them and are ordered among themselves in any convenient way. Of course, only a finite number of indeterminates are present and need be ordered at any step.

there is an $A_{2j}$ of order $t_2+k$ we let $B_{2k}$ be that $A_{2j}$. Otherwise $B_{2k}$ is defined as the remainder of the transform of $B_{2,k-1}$ with respect to the chain $B_{10}, \cdots, B_{1r}; B_{20}, \cdots, B_{2,k-1}$, where $r$ is chosen as the least integer such that no transform of $y_{10}$ occurring in $B_{20}, \cdots, B_{2,k-1}$ or the transform of $B_{2,k-1}$ is of order exceeding $t_1+r$. Proceeding similarly we let $B_{30}=A_{30}$. When $B_{30}, \cdots, B_{3,k-1}$ have been defined, we define $B_{3k}$ as the $A_{3j}$ of the proper order, if such exists, or as the remainder of the transform of $B_{3,k-1}$ with respect to $B_{10}, \cdots, B_{1s}$; $B_{20}, \cdots, B_{2r}$, where $s$ and $r$ are such that no transform of $y_{20}$ occurring in $B_{30}, \cdots, B_{3,k-1}$ or the transform of $B_{3,k-1}$ is of order exceeding $t_2+r$, and that no transform of $y_{10}$ occurring in these polynomials or in $B_{20}, \cdots, B_{2r}$ is of order exceeding $t_1+s$.

Continuing in this way we define the $B_{ij}$, $i=1, \cdots, p; j=0, 1, \cdots$. Each $B_{ij}$ is of order $t_i+j$ in $y_i$, and it is either a polynomial of the characteristic set of $\Lambda$ which is of this order in $y_i$ and free of $y_k$, $k>i$, or it is of the same degree in the $(t_i+j)$th transform of $y_i$ as is $B_{i,j-1}$ in the $(t_i-1+j)$th transform of $y_i$. Of course, $B_{ij}$ is free of $y_{kl}$, $k>i$.

Given an integer $r\geqq 0$ we let $s_p$ denote the maximum of $t_p$ and $r$. Let $r_p=s_p-t_p$. We then define $s_{p-1}$ to be the maximum of $t_{p-1}$, $r$, and the order of the highest transform of $y_{p-1}$ appearing in the polynomials $B_{p0}, \cdots, B_{pr_p}$, and let $r_{p-1}=s_{p-1}-t_{p-1}$. We define $s_{p-2}$ as the maximum of $t_{p-2}$, $r$, and the order of the highest transform of $y_{p-2}$ occurring in $B_{p-1,0}, \cdots, B_{p-1,r_{p-1}}; B_{p0}, \cdots, B_{pr_p}$. Continuing in this way we define successively $s_{p-3}, s_{p-4}, \cdots, s_1$ and let $r_i=s_i-t_i$, $i=1, \cdots, p$. Then

$$(2) \qquad B_{10}, \cdots, B_{1r_1}; \quad B_{20}, \cdots, B_{2r_2}; \cdots; \quad B_{p0}, \cdots, B_{pr_p}$$

is a chain. For $s$ such that no $u_{ij}$, $j>s$, occurs in (2) we define $\Lambda_{sr}$ as the prime p. i. (polynomial ideal[5]) in the indeterminates $u_{ij}$, $i=1, \cdots, q; j=0, 1, \cdots, s$, and $y_{km}$, $k=1, \cdots, p; m=0, 1, \cdots, s_k$, which consists of those polynomials of $\Lambda$ which involve only these $u_{ij}$ and $y_{km}$. Then (2) constitutes a characteristic set for $\Lambda_{sr}$ with $B_{ij}$ introducing $y_{i,t_i+j}$. The parametric indeterminates of $\Lambda_{sr}$ corresponding to this choice of characteristic set are those $u_{ij}$ occurring among its indeterminates and the $y_{km}$ with $m<t_m$. We note that all coefficients of the $B_{ij}$ are rational combinations of the coefficients appearing in (1) and their transforms.

Let $\lambda$ be any element of $\mathfrak{K}$. It will evidently suffice to show that $\lambda$ is in $\mathcal{G}$. We choose a positive integer $r$ such that $\lambda$ is in the field formed by adjoining to $\mathcal{J}$ the $\alpha_{ij}$, $i=1, \cdots, n$; $j=0, \cdots, r$. Let $s\geqq r$ be such that, with the $r$ just chosen, (2) is a characteristic set of

[5] Following Chapter IV of [2] we use this term to distinguish ideals of polynomials in the usual sense from difference ideals.

a prime p. i. $\Lambda_{sr}$. By the last remark of the preceding paragraph the coefficients of (2) are in $\mathcal{G}$. Since also $\mathcal{G} \subseteq \mathcal{K}$ it is readily seen that (2) is the characteristic set of a prime p. i. $\Pi$ with coefficients in $\mathcal{G}$ and involving the same indeterminates as $\Lambda_{sr}$. Similarly (2) is the characteristic set of a prime p. i. $\Pi'$ with coefficients in $\mathcal{G}(\lambda)$ and involving the same indeterminates as $\Lambda_{sr}$.

We obtain a generic zero of $\Lambda_{sr}$, $\Pi$, or $\Pi'$ by putting $u_{ij}=\beta_{ij}$; $y_{ij}=\gamma_{ij}$ for the appropriate ranges of the subscripts. We shall denote by $\delta_k$, where $k$ ranges over a suitable set of integers, those $\beta_{ij}$ and $\gamma_{ij}$ of the generic zero which have been equated to the $u_{ij}$ and $y_{ij}$ of the previously described set of parametric indeterminates of $\Lambda_{sr}$ (which are also, of course, a set of parametric indeterminates for either $\Pi$ or $\Pi'$). The remaining $\gamma_{ij}$ of the generic zero shall henceforth be denoted by $\epsilon_m$, where $m$ ranges over a suitable set of integers.

The degree of $\mathcal{G}(\delta_k, \epsilon_m)$ with respect to $\mathcal{G}(\delta_k)$ is given by the product of the degrees[6] of the polynomials of (2) in the indeterminates of $\Pi$ which they respectively introduce. We see in the same way that this product is the degree of $\mathcal{G}(\lambda)(\delta_k, \epsilon_m)$ with respect to $\mathcal{G}(\lambda)(\delta_k)$. But the fields $G(\lambda)(\delta_k, \epsilon_m)$ and $G(\delta_k, \epsilon_m)$ coincide because, by the stipulations concerning $r$ and $s$, $\lambda$ is in $\mathcal{G}(\delta_k, \epsilon_m)$. Hence the degrees of $\mathcal{G}(\delta_k, \epsilon_m)$ with respect to its two subfields $\mathcal{G}(\delta_k)$ and $\mathcal{G}(\lambda, \delta_k)$ are equal. Since these degrees are finite it follows that these subfields must be identical. In other words, $\lambda$ is in $\mathcal{G}(\delta_k)$.

We thus see that there exist elements $P$ and $Q$ in $\mathcal{G}[\delta_k]$, with $P$ not equal to zero, such that $P\lambda=Q$. Now the $\delta_k$ annul no nonzero polynomial with coefficients in $\mathcal{G}(\lambda)$. Hence the relation $P\lambda=Q$ must be an identity in the $\delta_k$. By equating coefficients of a suitable power product of the $\delta_k$ on both sides of this equation we find $p\lambda=q$, $p$ and $q$ in $\mathcal{G}$, and $p\neq 0$. Hence $\lambda$ is in $\mathcal{G}$. This completes the proof.

## References

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Rutgers University

[6] This follows from the work on pp. 89 and 90 of [2]. The inductive argument given there shows that a generic zero of $\Pi$ can be constructed by transcendental adjunctions followed by successive algebraic adjunctions of degrees equal to the degrees of the polynomials of the characteristic set in the indeterminates they introduce.