ON SYMMETRY IN CONVEX TOPOLOGICAL VECTOR SPACES

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A convex topological vector space E is called *symmetric* if and only if the topology of the strong bidual of E induces on E its given topology. A barrel in E is a closed, convex, equilibrated, absorbing set (see [1] for the terminology); E is called *barrelled* (French: espace tonnelé) if and only if every barrel is a neighborhood of 0. The properties of being symmetric and of being barrelled are but the two extreme examples of a family of properties which we first wish to discuss. Second, we give a new counter-example in the theory of convex topological vector spaces. All spaces considered are assumed Hausdorff.

1. If E and F are convex topological vector spaces, E' the topological dual of E, Σ a class of bound subsets of E such that $\bigcup[S|S\in\Sigma]=E$, then E_Σ' [respectively $\mathcal{L}_\Sigma(E, F)$] denotes the vector space E' [respectively $\mathcal{L}(E, F)$, the vector space of all continuous linear transformations from E into F] with the (convex) topology of uniform convergence on all members of Σ. For the important special case where Σ is all bound subsets, we write "b" for "Σ"; for that where Σ is all one-point subsets of E, we write "s" for "Σ." If E' is a total subspace of the algebraic dual of E, among all convex topologies on E yielding E' as dual there is a strongest, denoted by $\tau(E, E')$ [5, Theorem 5]; if a given convex topological vector space E with dual E' has the topology $\tau(E, E')$, it is called *relatively strong*.

If E is a convex topological vector space, E' its dual, Σ a class of bound subsets of E such that $\bigcup[S|S\in\Sigma]=E$, then E may be canonically identified (algebraically) with a subspace of the vector space (without topology) $(E_\Sigma')'$. Since $L\subseteq E'$ is equicontinuous if and only if the polar of L in E is a neighborhood of 0, it is easy to see that if Λ is a class of bound subsets of E_Σ' such that $\bigcup[L|L\in\Lambda]=E'$, then the topology induced on E by that of $(E_\Sigma')_\Lambda'$ is the given topology of E if and only if $(E_\Sigma')_\Lambda'=(E_\Sigma')_\Omega'$, where Ω is the class of all equicontinuous subsets of E'. Hence, since every equicontinuous subset of E' is bound in E_Σ', the topology of $(E_\Sigma')_b'$ always induces on E a stronger (i.e., at least as strong) topology than the given topology.

DEFINITION. E is Σ-*symmetric* if and only if the topology induced

Received by the editors June 7, 1954.

on E by that of $(E'_\Sigma)'_b$ is the given topology of E.

THEOREM 1. *If Λ is a class of bound subsets of E such that $\Lambda \supseteq \Sigma$ and if E is Σ-symmetric, then E is Λ-symmetric.*

PROOF. As the topology of E'_Λ is stronger than that of E'_Σ, there are fewer bound sets in E'_Λ than in E'_Σ; hence the topology of $(E'_\Lambda)'_b$ is weaker than the topology of $(E'_\Sigma)'_b$, and hence must also induce on E its given topology.

Theorem 1 shows that among all the properties of being Σ-symmetric, symmetry (i.e., b-symmetry) is the weakest. The strongest such property is s-symmetry (i.e., Σ-symmetry where Σ is the class of all one-point subsets of E). Theorem 2 shows that this property is precisely the property of being barrelled.

THEOREM 2. *Let E and F be convex topological vector spaces, F of nonzero dimension, Σ a class of bound subsets of E such that $\bigcup [S \mid S \in \Sigma] = E$. Then the following are equivalent*: (1) *E is Σ-symmetric.* (2) *The polars in E of all bound subsets of E'_Σ form a fundamental system of neighborhood of* 0 *for the topology of E.* (3) *Every bound subset of E'_Σ is equicontinuous.* (4) *Every bound subset of $\mathcal{L}_\Sigma(E, F)$ is equicontinuous.* (5) *Every barrel in E absorbing all members of Σ is a neighborhood of* 0. (6) *E is relatively strong, and every convex bound subset of E'_Σ has compact closure in E'_s.*

PROOF. The equivalence of (1), (2), and (3) follows immediately from our discussion above. The equivalence of (3) and (5) follows from the fact that a set is bound in E'_Σ if and only if its polar in E is a barrel absorbing all members of Σ. Proposition 2 of [**3**] asserts the equivalence of (5) and (6) for the special case of barrelled spaces, and Theorem 1 of [**3**] asserts that (5) implies (4) for barrelled spaces. In both cases an obvious modification of the proof yields the desired result. It remains to show that (4) implies (3). Let Ky_0 be a one-dimensional subspace of F, K the scalar field. $\phi: \lambda y_0 \rightarrow \lambda$ is a topological isomorphism from Ky_0 onto K. Let $L = [u \in \mathcal{L}(E, F) \mid u(E) \subseteq Ky_0]$ with the topology induced from $\mathcal{L}_\Sigma(E, F)$. It is immediate that ψ: $u \rightarrow \phi \circ u$ is a topological isomorphism from L onto E'_Σ. Let B be bound in E'_Σ, V a neighborhood of 0 in K. Then $Vy_0 = W \cap Ky_0$ where W is a neighborhood of 0 in F. $\psi^{-1}(B)$ is bound in L, hence in $\mathcal{L}_\Sigma(E, F)$, and hence is an equicontinuous subset of $\mathcal{L}(E, F)$. Therefore there exists a neighborhood W' of 0 in E such that if $u \in \psi^{-1}(B)$ then $u(W') \subseteq W$ and hence, as $u \in L$, $u(W') \subseteq W \cap Ky_0 = Vy_0$. But then if $v \in B$, $\phi^{-1} \circ v \in \psi^{-1}(B)$ so $\phi^{-1}(v(W')) \subseteq Vy_0$, i.e., $v(W') \subseteq \phi(Vy_0) = V$. Hence B is equicontinuous.

COROLLARY 1. *A necessary and sufficient condition that E be barrelled is that E be Σ-symmetric and that every bound subset of E'_s be bound in E'_Σ.*

"Sequentially complete" (i.e., all Cauchy sequences converge) can replace "complete" in Theorem 2 of [**3**]; hence

COROLLARY 2. *If every $S\in\Sigma$ is sequentially complete, then E is barrelled if and only if E is Σ-symmetric.*

It is obvious from Theorem 2 that for every theorem about barrelled spaces there is an analogue for Σ-symmetric spaces. We mention in particular the abstract version of the Banach-Steinhaus theorem, the proof of which for barrelled spaces is found in [**2**] or in Corollaries 1 and 2 of Theorem 1 of [**3**].

THEOREM 3. *Let E be Σ-symmetric, Φ a filter on $\mathcal{J}(E, F)$, the vector space of all functions from E into F, $u_0\in\mathcal{J}(E, F)$. Then $u_0\in\mathcal{L}(E, F)$ and Φ converges to u_0 in the topology of uniform convergence on all precompact subsets of E under any of the following additional assumptions: (1) Φ contains a bound subset of $\mathcal{L}_\Sigma(E, F)$ and Φ converges pointwise to u_0; (2) F is quasi-complete (i.e., every closed, bound subset of F is complete), Φ contains a bound subset of $\mathcal{L}_\Sigma(E, F)$, and Φ converges pointwise to u_0 on a total subset of E.*

2. E is called *semi-reflexive* if $(E'_b)'=E$ (algebraically). E is called *boundedly closed* if every bound linear functional on E is continuous. By Theorem 2, a symmetric space is relatively strong. We show the converse is false by giving an example of a semi-reflexive, relatively strong space F whose strong dual F'_b is a Banach space, but which is neither symmetric nor boundedly closed.

Let E be a nonreflexive Banach space (e.g., L^1 of the unit interval). E may be regarded as a total subspace of the algebraic dual of E'; we let F be the vector space E' together with the convex topology $\tau(E', E)$. F is thus by definition relatively strong. We show $F'_b=E$ (algebraically and topologically). Since E is barrelled, the classes of all bound subsets of E'_s, of all bound subsets of E'_b, and of all equicontinuous subsets of E' are identical (Theorem 2); this class is also identical with the class of all bound subsets of F, since E, the topological dual of F, is also the topological dual of E'_s [**4**, Theorem 2], and hence F and E'_s have the same bound subsets [**5**, Theorem 7]. V is a neighborhood of 0 in F'_b if and only if V contains the polar of a bound subset of F; V is a neighborhood of 0 in E if and only if V contains the polar of an equicontinuous subset of E'; hence E is topologically and algebraically identical with F_b'. Thus $(F_b')' = E' = F$, so F is semi-reflexive. As E is not reflexive, $(E_b')'$ strictly contains E. The topology of $(F_b')_b' = E_b'$ is thus strictly stronger than that of F since F and $(F_b')_b'$ have different topological duals; hence F is not symmetric. Also, as the bound subsets of F and of E_b' coincide, every linear functional in $(E_b')'$ is bound on F, but as $F' = E \neq (E_b')'$, there exist bound linear functionals on F which are not continuous. Hence F is not boundedly closed.

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