A REMARK ON FINITELY GENERATED NILPOTENT GROUPS

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In this note we use the word nilpotent in the strong sense; a group G is nilpotent if its lower central series, defined by $H_0=G$ and $H_{i+1}=(H_i, G)$, terminates in the identity in a finite number c of steps, and c is then the class of G. As usual, G^n denotes the subgroup of G generated by the nth powers of elements of G. Our object is to prove the following result.

THEOREM 1. *If G is a finitely generated nilpotent group, then the intersection of the groups G^p, for any infinite set of primes p, is finite.*

We recall first some well known facts about finitely generated nilpotent groups. They will be found, in essence, in Hall [2] and Hirsch [3]. First, if X is a subset of G that generates G modulo its derived group H_1, then H_{i-1} is generated modulo H_i by the left normed commutators

$$(\cdots((x_1, x_2), x_3), \cdots, x_i), \qquad x_r \in X. \tag{1}$$

Thus H_{i-1} is finitely generated modulo H_i. In particular, H_{c-1} is a finitely generated abelian group, and its subgroups therefore satisfy the maximal condition. By induction on c, so do those of G. Again, if any of x_r is of finite order modulo H_1, then the commutator (1) is of finite order modulo H_i. Hence finitely many elements of G of finite order generate a finite subgroup, and it follows from the maximal condition on the subgroups of G that in fact there are only a finite number of elements of finite order in G. In particular, there are only a finite number of primes p for which G contains elements of order p. Lastly, for a prime q greater than the class c of G, Hall's formula becomes

$$(xy)^q = x^q y^q z_1^q \cdots z_r^q,$$

where each z_j is a commutator in x and y. It follows easily, by backward induction on the weight of y (the least integer i such that y does not belong to H_i), that for some u in G, $x^q y^q = u^q$. That is, G^q is not merely generated by, but consists of the qth powers of elements of G.

We can now prove Theorem 1. If the class c of G is 1, so that G is abelian, the theorem follows immediately from the basis theorem for finitely generated abelian groups. We therefore use induction on c, and

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assume $c>1$. Let K be the intersection of an infinite set of subgroups G^p. By the hypothesis of the induction, KH_{c-1}/H_{c-1} is finite, and hence so is the isomorphic group $K/(K\cap H_{c-1})$. It is sufficient therefore to prove that $K\cap H_{c-1}$ is finite. If $y\in K$, then surely $y\in G^q$ for an infinity of primes q with $q>c$; thus for an infinity of q there exists y_q such that $y_q^q=y$. If also $y\in H_{c-1}$, the cyclic group $\{y\}$ is normal, and unless $y_q\in\{y\}$, the factor group $G/\{y\}$ contains an element of order q. Since this is possible for only a finite set of primes q, for some q, $y_q\in\{y\}$, whence y is of finite order. That is, $K\cap H_{c-1}$ contains only elements of finite order, and is therefore a finite group. This concludes the proof.

It is perhaps of interest to remark that Theorem 1 yields a short proof of the following theorem of Baer [1].

THEOREM 2. *There is an integer n such that the intersection of all characteristic subgroups of G whose indices are prime powers p^a with $a\leqq n$ is the identity.*

For, by a theorem of Hirsch [4], G (or, indeed, any soluble group with the maximal condition for subgroups) has a subgroup N of finite index which contains no elements of finite order. We can take this subgroup to be characteristic. For, if its index is h, we can replace it by the intersection of all subgroups of G of index h, which (Baer, loc. cit.) is still of finite index. Then G/N is a finite nilpotent group, and so the direct product of its Sylow subgroups, whence N is the intersection of a finite set of characteristic subgroups of G, whose indices are prime powers. If to these we add any infinite set of subgroups G^p, we obtain a set whose intersection is the identity. For this intersection contains no element of finite order by choice of N, and none of infinite order by Theorem 1. All the groups are characteristic subgroups of G of prime power index p^a, and to show that the exponents a are bounded, we may concentrate on the groups G^p, since the others are a finite set only. But if the number of generators of H_{i-1}/H_i is $r(i)$, the index of G^p is p^a with $a\leqq r(1)+r(2)+\cdots+r(c)$.

REFERENCES

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4. ———, *On infinite soluble groups* III, ibid. vol. 49 (1945–47) pp. 184–194.

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