REDUCIBILITY OF POSITIVE TYPE POLYNOMIALS

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1. Introduction. Let T_+ denote the set of all positive type polynomials, i.e., $T_+ = \{ \sum_{j=0}^{m} a_j x^j \} = \{ P_m(x) \}$ where $a_j \geq 0$, $0 \leq j \leq m \geq 1$, $a_m \neq 0$. Clearly, T_+ is closed under multiplication but not factorization. An analogous but wider class of functions has been considered by Rosenbloom [4].

Let T_i be that subset of T_+ which is irreducible over T_+ (i.e., $P_m(x) \in T_i$, $P_m(x) = P_k(x) \cdot P_{m-k}(x)$, $0 < k < m$, implies $P_k \notin T_+$ or $P_{m-k}(x) \notin T_+$). It would be of interest to have a criterion for deciding when an arbitrary member of T_+ lies in T_i or $T_r = T_+ - T_i$.

For $m = 2$, the sign of the discriminant of the quadratic provides the answer while for $m = 3$ a NSC that $P_m(x) \in T_r$ is $a_1 a_2 \geq a_0 a_3$. For $m = 4$ a NSC can be given (§4) in terms of the location of a root of $P_4(-x)$ (which is not very satisfactory) from which simpler conditions can be derived in certain cases. Some partial results are given for general m.

The preceding is manifestly applicable to the so-called "arithmetic of probability distributions" (initiated by Paul Lévy, see e.g. [2]) provided the random variables take on only a *finite* number of rational values (or values of the form $a + kb$, k rational, a, b real). The proof of Theorem 1 contains a method of constructing an indecomposable (prime) distribution for any positive integer $m \geq 2$.

2. Preliminary considerations. Unless the contrary is stated it will be understood that $P_m(x) \in T_+$. In the treatment of this polynomial there is no loss of generality in supposing $a_m = 1$, $a_0 > 0$. Denoting by A_i, A_i', B_i, C_i positive real numbers, the canonical decomposition of $P_m(x) \in T_+$ into quadratic and linear factors can only be of the form

$$\prod_j (x^2 - A_j x + B_j) \prod_k (x^2 + A_k' x + B_k) \prod_i (x^2 + B_i) \prod_n (x + C_n)$$

$$= \left(\prod_j Q_j^- \right) \left(\prod_k Q_k^+ \right) \left(\prod_i Q_i^0 \right) \left(\prod_n L_n^+ \right).$$

The number of linear factors L_n^+ may be supposed to be at most one.

Let $T_q \subset T_+$ denote the set of positively quadral polynomials (see [3]), i.e., all quadratic factors are of the Q^+ or Q^0 variety and any linear factors are of the form L^+. Hurwitz [1] has given a NSC that all roots of $P_m(x)$ have negative real part. By inserting a few

equality signs here and there in I. Schur's proof of Hurwitz's Theorem (see [5, appendix]) one obtains the following elementary extension:

EXTENDED HURWITZ THEOREM. *A necessary and (if $a_1 \neq 0$) sufficient[1] condition that $P_m(x) \in T_q$ is that the determinants*

$$D_1 = a_1, \qquad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \cdots, D_m = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{2m-1} & a_{2m-2} & & \cdots & a_m \end{vmatrix}$$

be non-negative. Here $a_j = 0$ for $j > m$.

Obviously, T_q is closed under factorization and $T_q \subset T_r$. Thus, the problem originally stated retains interest only if the extended Hurwitz criterion is violated, i.e., $P_m(x) \in T^* = T_+ - T_q$. In such a case there is at least one factor of the Q^- type and this may be represented by $Q^-(x, \epsilon) = x^2 - Ax + \epsilon A^2$ where ϵ, $A > 0$. Actually $\epsilon > 1/4$ if $Q^-(x, \epsilon)$ is to have no positive root (be a factor of $P_m(x)$).

3. Minimal degree polynomials.

THEOREM 1. *A NSC that there exist $P_m(x) \in T_+$ containing $Q^-(x, \epsilon)$ $= x^2 - Ax + \epsilon A^2$ as a factor is that $m \geq$ smallest integer j such that j arc cos $(1/2\epsilon^{1/2}) \geq \pi$.[2]*

Necessity. By the transformation $x = Ay$, we may suppose $A = 1$. Let $\delta = 4\epsilon - 1 > 0$ and $b_j = 2^{-j} a_j \geq 0$. As the roots of $Q^-(x, \epsilon) = 0$ are also roots of $P_m(x) = 0$,

$$\text{(1.1)} \qquad \sum_{j=0}^{m} b_j \left[\sum_{k=0}^{[j/2]} \binom{j}{2k} (-1)^k \delta^k \right] = 0,$$

$$\text{(1.2)} \qquad \delta^{1/2} \sum_{j=1}^{m} b_j \left[\sum_{k=0}^{[(j-1)/2]} \binom{j}{2k+1} (-1)^k \delta^k \right] = 0.$$

Denote the coefficients of b_j in (1.1) by $f_j = f_j(\delta) = 2^{j-1} \bar{f}_j(\epsilon)$ and those in (1.2) by $h_j = \delta^{1/2} g_j = \delta^{1/2} g_j(\delta) = \delta^{1/2} 2^{j-1} \bar{g}_j(\epsilon)$. These quantities are virtually the Tchebycheff polynomials of the 1st and 2nd kind,

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[1] If $a_1 = 0$ and $a_{2j+1} > 0$ for some $j \geq 1$, $P_m \notin T_q$; if $a_{2j-1} = 0$ for all $j \geq 1$, $P(x) \in T_q$ is of even degree and $P_{2m}(x) = R_m(x^2) = R_m(y)$ whence if $R_m(y) \notin T_q$, $P_{2m}(x) \notin T_q$.

[2] The principal value of the arc cosine is to be taken.

say $T_j^{(1)}(x)$ and $T_j^{(2)}(x)$. In fact $T_j^{(1)}(x) = x^i f_j((1-x^2)/x^2)$ and $T_j^{(2)}(x) = x^j g_{j+1}((1-x^2)/x^2)$. Letting $1 + i\delta^{1/2} = r$ (cos $\theta + i$ sin θ) with $r = (1+\delta)^{1/2}$, $0 < \theta < \pi/2$ we have from DeMoivre's theorem

(1.3)
$$f_j = f_j(\delta) = (1+\delta)^{i/2} \cos j\theta,$$
$$h_j = h_j(\delta) = (1+\delta)^{i/2} \sin j\theta = \delta^{1/2} g_j(\delta).$$

Rewrite (1.1) and (1.2) as

(1.4, 1.5)
$$\sum_{j=0}^{m} b_j f_j = 0 = \sum_{j=1}^{m} b_j h_j.$$

Since $b_j \geqq 0$ (actually $b_0 > 0$), from (1.4) and (1.5) there must be at least one non-negative f_i and non-positive h_i for $i \leqq m$. By (1.3) this requires $j\theta \geqq \pi$, concluding necessity.

For any given ϵ ($> 1/4$), denote by $M = M(\epsilon) \geqq 3$ the minimal (integral) value of $\bar{m}$ for which $\bar{m}$ arc cos $(1/2\epsilon^{1/2}) \geqq \pi$. Then $P_M(x)$ exists and will be called a minimal polynomial containing Q^- or minimal for Q^- or simply "minimal for ϵ." As $\epsilon \rightarrow 1/4$, the roots of $Q^-(x, \epsilon)$ approach the positive real axis and $M(\epsilon) \rightarrow \infty$. $M(\epsilon)$ is a step function continuous on the right and may be computed from the relation $\epsilon \geqq (1/4) \sec^2 (\pi/M)$.

Sufficiency. It suffices to construct a $P_M(x)$. From (1.1), $f_0 = 1$. Choose $b_M = 2^{-M}$, $b_{M-1} = 2^{-M}|h_M|(h_{M-1})^{-1}$, and $b_1 = b_2 = \cdots = b_{M-2} = 0$ thus satisfying (1.5). To fulfill (1.4) take $b_0 = 2^{-M}[|f_M| - f_{M-1}|h_M|(h_{M-1})^{-1}]$. From (1.3), $h_{M-1} > 0$, $f_{M-1} < 0$ whence $b_0 > 0$. If $A \neq 1$, $a_j = 2^i A^{-i} b_j \geqq 0$. Q.E.D.

Clearly, the minimal degree M although not $P_M(x)$ is unique.[3] A less accurate but simpler bound is given by

COROLLARY 1.1. *For $1/4 < \epsilon \leqq 1/2$, a necessary condition that $Q^-(x, \epsilon)$ divides $P_m(x)$ is $m \geqq M \geqq$ smallest integer j such that $j \geqq \pi \delta^{-1/2}$.*

PROOF. This follows from arc tan $\delta^{1/2} < \delta^{1/2}$ for $0 < \delta < 1$.

Using the trigonometric identities for sin $(\theta + \phi)$ and cos $(\theta + \phi)$ and (1.1), (1.2) we have $f_j(\delta) = f_{j-1}(\delta) - \delta g_{j-1}(\delta)$ and $g_j(\delta) = g_{j-1}(\delta) + f_{j-1}(\delta)$, $j \geqq 2$, from which it is readily deduced that both $g_j(\delta)$ and $f_j(\delta)$ satisfy the equation $g_j(\delta) = 2g_{j-1}(\delta) - (1+\delta)g_{j-2}(\delta)$, $j \geqq 2$. A more usable form of this equation is (see just below (1.2))

(1.6)
$$\bar{g}_j(\epsilon) = \bar{g}_{j-1}(\epsilon) - \epsilon \bar{g}_{j-2}(\epsilon), \qquad\qquad j \geq 2.$$

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[3] In the special cases $\epsilon = \epsilon_M^* = 4^{-1} \sec^2 (\pi/M)$, $M = 3, 4, \cdots$, the minimal polynomial *is* unique and of the form $x^M + a_0$. For from (1.5), $h_M = 0$ whence also $b_j = 0$, $1 \leqq j \leqq M-1$.

Define now a "residual polynomial"

$$P_{m-2}(x) = \sum_{j=0}^{m-2} c_j x^i \text{ by } P_m(x) = \sum_{j=0}^{m} a_j x^i = (x^2 - Ax + \epsilon A^2)P_{m-2}(x).$$

We have then

THEOREM 2. *If $Q^-(x, \epsilon)$ divides the minimal polynomial $P_M(x) \in T_+$, the corresponding residual polynomial $P_{M-2}(x) \in T_+$ and (excluding $c_{M-2} = a_M = 1$) its coefficients form an increasing sequence.*

PROOF. Consider at first any $P_m(x) \in T_+$ and as before let $A = 1 = a_m = c_{m-2}$. If we define $c_{-2} = c_{-1} = c_{m-1} = c_m = 0$, then for any residual polynomial (regardless of minimality)

$$(2.1) \qquad\qquad \epsilon c_k - c_{k-1} + c_{k-2} = a_k, \qquad k = 0, 1, \cdots, m.$$

By induction using (1.6) it follows that

$$(2.2) \qquad\qquad \epsilon^{j+1} c_j = \sum_{i=0}^{j} \epsilon^{i-i} a_{j-i} \bar{g}_{i+1}(\epsilon), \quad j = 0, 1, \cdots, m - 2.$$

If now $m = M(\epsilon)$, then $c_j \geqq 0$, $j = 1, 2, \cdots, M-2$, since $\bar{g}_k > 0$ for $k < M$ in view of (1.3). Next from (2.2) and subsequently (1.6),

$$c_{j+1} - c_j = \epsilon^{-1} a_{j+1} + \sum_{i=0}^{j} \epsilon^{-(i+2)} [\bar{g}_{i+2} - \epsilon \bar{g}_{i+1}] a_{j-i}$$

$$= \epsilon^{-1} a_{j+1} + \sum_{i=0}^{j} \epsilon^{-(i+2)} a_{j-i} \bar{g}_{i+3} \geqq \epsilon^{-(j+2)} a_0 \bar{g}_{j+3} \geqq 0$$

so long as $j < M(\epsilon) - 3$.

COROLLARY 2.1. *If $Q^-(x, \epsilon)$ divides $P_m(x) \in T_+$ and $m \leqq 2M - 1$, the residual polynomial $P_{m-2}(x) \in T_+$ and its coefficients from c_0 to c_{M-3} form an increasing sequence.*

PROOF. Suppose $P_{n+r}(x) = (x^2 - x + \epsilon) \sum_{j=0}^{n+r-2} c_j x^i$. As before $c_0, c_1, \cdots, c_{M-2}$ are non-negative. By a backwards induction using (2.1) and (1.6), $c_{n+j-1} = \sum_{i=j+1}^{r} \bar{g}_{i-j}(\epsilon) a_{n+i}$ for $j = 0, 1, \cdots, r-1$. The largest subscript of $\bar{g}$ occurs for $i = r$, $j = 0$ giving $\bar{g}_r(\epsilon)$. Hence if $n = M$ and $r \leqq M - 1$ the coefficients $c_{M-1}, c_M, \cdots, c_{M+r-2}$ are non-negative. Thus $P_{m-2}(x) \in T_+$ for $m \leqq 2M - 1$. Q.E.D.

If $m \geqq 2M$, $P_{m-2}(x)$ need not $\in T_+$; e.g., if $M(\epsilon) = M(2) = 3$

$$(x^6 + 5x^3 + 8) = (x^2 - x + 2)(x^4 + x^3 - x^2 + 2x + 4).$$

Denote by $M_i = M_i(\epsilon_i)$ the degree of a minimal polynomial contain-

ing $Q_i^-(x, \epsilon_i)$ as a factor. We consider next some effects of having two Q^- factors.

THEOREM 3. *Let* $P_m(x) \in T_+$. *If* $P_m(x) = (x^2 - A_1x + \epsilon_1 A_1^2)P_{m-2}(x)$ *with* $m = M_1 + j$ *where* $j = 0$ *or* 1 *and* $P_{m-2}(x) = (x^2 - A_2x + \epsilon_2 A_2^2)P_{m-4}(x)$ *where necessarily* $\epsilon_1 \leq \epsilon_2$, *then* $M_1 > 2M_2 - 1 - j$ *and* $P_{m-2}(x)$ *is not minimal for* ϵ_2.

PROOF. Let $P_m(x) = (x^2 - A_2x + \epsilon_2 A_2^2)R_{m-2}(x)$. As $m \leq M_1 + 1$, $R_{m-2}(x) \notin T_+$. By Corollary 2.1, $M_1 + j > 2M_2 - 1$. Furthermore $P_{m-2}(x)$ cannot be minimal for ϵ_2 since $M_1 + j - 2 = M_2$ would imply $3 > M_2$. Q.E.D.

THEOREM 4. *If* $P_6(x) = \sum_{j=0}^{6} a_jx^j$ ($\in T_+$) *is minimal for* $Q^-(x, \epsilon_1)$, *then the residual polynomial* $P_4(x) \in T_q$.

PROOF. For suppose $P_6(x) = (x^2 - Cx + \epsilon_1 C^2)P_4(x)$, $P_4(x) = (x^2 - Ax + \epsilon A^2)(x^2 + Bx + D)$ with positive C, A, D, ϵ. As $a_5 \geq 0$ implies $B > 0$, let $D = \mu B^2$, $\mu > 0$. As before, take $A = 1$. By Theorem 2

$$(4.1, 4.2, 4.3) \quad B - 1 \geq \epsilon + \mu B^2 - B \geq B(\epsilon - \mu B) \geq \epsilon \mu B^2.$$

By Theorem 3, P_4 is not minimal for ϵ; hence $M(\epsilon) = 3$, $\epsilon \geq 1$. Now (4.1) implies $\mu^{-1} - \epsilon - 1 \geq \mu(B - \mu^{-1})^2 \geq 0$, i.e., $\mu^{-1} - 1 \geq \epsilon \geq 1$ and in particular $\mu \leq 1/2$. From (4.2), $g(B) = 2\mu B^2 - (1 + \epsilon)B + \epsilon \geq 0$. In view of $\mu \leq 1/2$, $g(B) = 0$ has two real roots $B_1 \leq B_2$. Again $\mu \leq 1/2$ insures $B_1 < 1$. As $P_6(x) \in T_+$ implies $B \geq C + 1 \geq 1$, we must have $B \geq B_2$.

On the other hand from (4.3), $B \leq \epsilon \mu^{-1}(1 + \epsilon)^{-1}$. This contradicts $B \geq B_2$ in view of $(1 + \epsilon)(4\mu)^{-1} \geq \epsilon \mu^{-1}(1 + \epsilon)^{-1}$. Q.E.D. This theorem is not true for general m. For example, let arc cos $(2\epsilon^{1/2})^{-1} = 18°$, $M(\epsilon) = 10$. Then

$$(x^{10} + 1) = (x^2 + 1)(x^2 - 2x \cos 18° + 1)(x^2 - 2x \cos 54° + 1)$$
$$\cdot (x^2 + 2x \sin 36° + 1)(x^2 + 2x \sin 72° + 1).$$

The preceding suggests

CRITERION 1. Given $P_m(x) \in T^* = T_+ - T_q$ (verifiable by the extended Hurwitz theorem), find its canonical decomposition and the ϵ_k corresponding to all quadratic factors of the type $Q_k^-(x, \epsilon_k) = x^2 - A_kx + \epsilon_k A_k^2$ where $A_k > 0$. Let $\epsilon = \min_k \epsilon_k$. If $M(\epsilon) = m$, $P_m(x) \in T_i$. Also if $P_m(x)$ has no real roots and $M(\epsilon) > m - 2$, $P_m(x) \in T_i$.[4]

4. **Special cases.** For $m = 3$, the extended Hurwitz criterion yields $a_1a_2 < a_0a_3$ as a NSC for $P_3(x) \in T_i$.

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[4] An obvious sufficient condition that $P_m(x) \in T_i$ is that all differences of pairs of exponents with nonzero coefficients a_j be distinct [2].

Consider next the case $m=4$, $a_m=a_4=1$. Then if $a_1 \neq 0$, a NSC that $P_4(x) \in T^*$ is $a_3[a_1 \cdot a_2 - a_0 \cdot a_3] < a_1^2$. Should $a_1 = 0$, then $a_3 > 0$ or $a_3 = 0$, $a_2^2 < 4a_0$ is NS that $P_n(x) \in T_i \subset T^*$.

Define $Q_2(x) = x^2 - a_3 \cdot x + a_2$ with $r_1 \leq r_2$ the roots of $Q_2(x) = 0$. Then if $T_r^* = T_r - T_q$, we have

THEOREM 5. *Let $P_4(x) \in T^*$. If $a_3^2 \geq 4a_2$, $(<)$, a NSC for $P_4(x) \in T_r^*$ is that $P_4(-x)$ have a root in $(0, r_1]$ or $[r_2, a_3]$, $(in \ (0, a_3])$.*

PROOF. The only possible factorization is $P_4(x) = (x+z)[x^3 + (a_3 - z)x^2 + Q_2(z)x + Q_3(z)]$ where $Q_3(z)$ is a cubic polynomial and $P_4(-z) = 0$, $0 < z \leq a_3$, $Q_2(z) \geq 0$. Also if these conditions obtain,

$$P_4(x) \in T_r^*.$$

Theorem 5 is hardly a satisfactory answer for the case $m=4$ as the criterion involves a 4th degree equation. However, it leads to the more cumbersome but intrinsically simpler conditions of the following theorems.

THEOREM 6. *Let $P_4(x) \in T^*$ and $a_3^2 < 4a_2$. If $a_3 < a_0 a_1^{-1}$ or $a_2 - 4^{-1}a_3^2 > 4^{-1}a_1^2 a_0^{-1}$, $P_4(x) \in T_i$. Further if $a_0 a_1^{-1} < 2^{-1}a_3$ and $a_2 - 4^{-1}a_3^2 \leq 2a_1 a_3^{-1} - 4a_0 a_3^{-2}$, $P_4 \in T_r^*$ and there are two factorizations or one according as*

$$(5.1) \qquad\qquad a_2 \geq a_1 a_3^{-1} - a_0 a_3^{-2}$$

is or is not valid.

THEOREM 7. *Let $P_4(x) \in T^*$ and $a_3^2 \geq 4a_2$. If $a_3 < a_0 a_1^{-1}$, $P_4(x) \in T_i$.*

(i) *If $2^{-1}a_3 > a_0 a_1^{-1}$, a NSC for two, one, or no factorizations of $P_4(x)$ within T^+ is that both, just one, or neither of (5.1) and $a_0^2 a_1^{-2} + a_2 \geq a_3 \cdot a_0 \cdot a_1^{-1}$ hold.*

(ii) *If $r_2 > a_0 a_1^{-1} \geq 2^{-1}a_3$, $P_4(x) \in T_r^*$ (with just one factorization) or T_i according as (5.1) holds or not.*

(iii) *If $r_2 = a_0 a_1^{-1} \geq 2^{-1}a_3$, then $P_4(x) \in T_r^*$ and there are two factorizations or one according as both (5.1) and $a_1^4 > a_0^2[2a_0 - a_1 a_3]$ hold or not.*

PROOF OF THEOREMS 6 AND 7. By Theorem 5, a NSC that $P_4(x) \in T_r^*$ is that there exist z in $(0, a_3]$ such that $z^{-2}P_4(-z) = 0$ and $Q_2(z) \geq 0$. But $z^{-2}P_4(-z) = Q_2(z) - R(z)$ where $R(z) = a_1 z^{-1} - a_0 z^{-2}$. Hence, it is NS that $Q_2(z) = R(z) \geq 0$ for some z in $(0, a_3]$. A consideration of the behavior of the graphs of these two functions yields the particularized results. The condition in (iii) arises from an examination of the slopes at $z = a_0 a_1^{-1}$. Q.E.D.

The relationships among the coefficients a_j are not exhausted by the considerations of the last two theorems. The remaining cases can

be dealt with but the conditions obtained would be no easier to verify than that of Theorem 5.

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