NOTE ON THE FIRST CESÀRO MEAN OF THE DERIVED CONJUGATE SERIES OF FOURIER SERIES

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1. Let $f(t)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π and let

$$(1.1)\quad f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_{1}^{\infty} A_n(t).$$

The conjugate series of (1.1) at $t=x$ is

$$(1.2)\quad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{1}^{\infty} B_n(x).$$

Then the differentiated series of (1.2) is

$$(1.3)\quad -\sum_{n=1}^{\infty} n(a_n \cos nx + b_n \sin nx) = -\sum_{1}^{\infty} nA_n(x) = -\sum_{1}^{\infty} L_n, \text{ say.}$$

We write $\phi(t) = f(x+t) + f(x-t) - 2f(x)$, $h(t) = \phi(t)/t - d$, where $d \equiv d(x)$. Let t_n be the nth Cesàro mean of order 1 of the series (1.3). The object of the present note is to prove the following

THEOREM. *If*[1]

$$(1.4)\quad \int_t^{\pi} \frac{|h(u)|}{u} du = o\left(\log \frac{1}{t}\right), \qquad \text{as } t \to 0,$$

then

$$t_n/\log n \to d/\pi, \qquad \text{as } n \to \infty.$$

The relation between Condition (1.4) and the Condition

$$(1.5)\quad \int_0^t |h(u)| du = o(t)$$

is brought out by the following

LEMMA. *If* $\int_0^t |h(u)| du = o(t)$, *as* $t \to 0$, *then*

$$\int_t^{\pi} \frac{|h(u)|}{u} du = o\left(\log \frac{1}{t}\right).$$

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[1] Throughout the present note it is assumed that $h(t)$ is L in $(0, \pi)$.

On the other hand, if $\int_t^\pi (|h(u)|/u)du = o(\log(1/t))$ *then*

$$\int_0^t |h(u)|\,du = o\left(t\log\frac{1}{t}\right).$$

This is known.[2]

If $k_2(n, t)$ denotes the nth Cesàro mean of order 2 of the sequence $2\pi^{-1}\sin nt$, then we have the following inequalities:[3]

$$\left|\frac{d}{dt}\,k_2(n, t)\right| \begin{cases} \leqq An, & (1.6)\\ \leqq An^{-1}t^{-2}. & (1.7)\end{cases}$$

(1.6) is trivial, and (1.7) is known.[4]

2. Proof of the Theorem. Now

$$\begin{aligned} L_k = kA_k(x) &= \frac{k}{\pi}\int_0^\pi \phi(t)\cos ktdt \\ &= \frac{k}{\pi}\int_0^\pi \frac{\phi(t)}{t}\,t\cos ktdt = \frac{k}{\pi}\int_0^\pi \{h(t) + d\}\,t\cos ktdt \qquad (2.1)\\ &= \frac{1}{\pi}\int_0^\pi th(t)k\cos ktdt + \frac{d}{\pi}\int_0^\pi tk\cos ktdt \\ &= \beta_k + \gamma_k, \quad \text{say.}\end{aligned}$$

Integrating by parts, we get

$$\gamma_k = -\frac{d}{\pi}\,\frac{1-(-1)^k}{k} = -\frac{d}{\pi}\,\omega_k, \text{ say.} \qquad (2.2)$$

Hence

$$\begin{aligned} -t_n &= \frac{1}{n+1}\sum_{k=1}^n (n-k+1)L_k = \frac{1}{n+1}\sum_{n+1}^n (n-k+1)\beta_k \\ &\quad + \frac{1}{n+1}\sum_1^n (n-k+1)\gamma_k = t_n^{(1)} + t_n^{(2)}, \text{ say.}\end{aligned} \qquad (2.3)$$

Now

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[2] M. L. Misra, Quart. J. Math. Oxford Ser. vol. 18 (1947) p. 149 with ϕ replaced by h.

[3] A is used to denote a number independent of n and t; but its value may differ from occurrence to occurrence.

[4] N. Obreschkoff, Bull. Soc. Math. France vol. 62 (1934) pp. 167–168.

$$
\begin{aligned}
t_n^{(1)} &= \frac{1}{n+1}\,\frac{1}{\pi}\int_0^{\pi} th(t)\left(\frac{d}{dt}\right)\sum_1^n (n-k+1)\sin ktdt \\
&= \frac{n+2}{4}\int_0^{\pi} th(t)\left(\frac{d}{dt}\right)k_2(n,t)dt \qquad (2.4)\\
&= \frac{n+2}{4}\left\{\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right\} \\
&= l_1 + l_2, \text{ say.}
\end{aligned}
$$

Using the lemma, (1.6), and (1.7), it is easy to see that

$$
l_1 = o(\log n),
$$
$$
l_2 = o(\log n),
$$

so that

$$
t_n^{(1)} = o(\log n). \tag{2.5}
$$

Lastly we consider $t_n^{(2)}$. By (2.2) and (2.3), we have

$$
t_n^{(2)} = -\frac{d}{\pi}\,\frac{1}{n+1}\sum_{k=1}^{n}(n-k+1)\omega_k. \tag{2.6}
$$

By partial summation we find[5]

$$
\begin{aligned}
&\frac{1}{n+1}\sum_{k=1}^{n}(n-k+1)\omega_k \\
&= \frac{1}{n+1}\sum_{k=1}^{n}\Omega_k \\
&= \frac{1}{n+1}\sum_{1}^{n}(\log k + C + E_k) \\
&= \frac{1}{n+1}\log\Gamma(n+1) + C + o(1) \qquad (2.7)\\
&= \frac{1}{n+1}\left\{\left(n+\frac{1}{2}\right)\log(n+1) - (n+1) + C_1 + o(1)\right\} \\
&\quad + C + o(1)^{6} \\
&\sim \log n.
\end{aligned}
$$

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[5] Here $\Omega_k = \sum_{r=1}^{k}\omega_r = \log k + C + E_k$, where C is a constant and $E_k \to 0$ as $k \to \infty$.

[6] E. C. Titchmarsh, *Theory of functions*, 2d. ed., 1939, p. 57.

Thus by (2.6) and (2.7)

$$t_n^{(2)} \sim -(d/\pi) \log n. \tag{2.8}$$

Hence by (2.3), (2.5), and (2.8), we have

$$t_n \sim (d/\pi) \log n,$$

which completes the proof.[7]

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[7] A note dealing with the analogous result for the differentiated Fourier series proved under the condition corresponding to (1.5) has been accepted by this journal. This condition may be replaced by the one corresponding to (1.4).

ON SUFFICIENT CONDITIONS FOR OVERCONVERGENCE[1]

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Ostrowski has shown that a Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with finite nonzero radius of convergence overconverges in a neighborhood of each point of the circle of convergence at which $f(z)$ is regular if $a_n = 0$ for $m_k < n < n_k$, m_k, $n_k \to \infty$ as $k \to \infty$, $m_k < \lambda n_k$, $\lambda < 1$, $k = 1, 2, 3, \cdots$ (i.e. the series has Hadamard gaps), while every overconvergent series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ can, in turn, be written in the form $f(z) = g(z) + h(z)$ where the Taylor series $g(z)$ has Hadamard gaps and $h(z)$ is a Taylor series whose radius of convergence is larger than that of $f(z)$. For a summary of results on overconvergence up to 1937 including proofs of Ostrowski's theorems the reader is referred to [1]. It is our purpose in this work to consider some conditions on the distribution of the zeros of the polynomials that stand between the gaps of a "gap series" that are sufficient to imply the overconvergence of the series, and to make an application of such results to a problem in the Fatou-Pólya circle of ideas.

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