THE POSITION OF C-SETS IN SEMIGROUPS[1]

A. D. WALLACE

A *continuum* is a compact connected Hausdorff space. If X is a continuum, then $C\subset X$ is a *C-set* if (i) $C\neq X$, (ii) for each subcontinuum $A\subset X$ that meets C we have either $A\subset C$ or $C\subset A$. Thus a composant of a solenoid is a C-set.

A *mob* is a Hausdorff space together with a continuous associative multiplication. A *clan* is a compact connected mob with (two-sided) unit, denoted by u. *In what follows we assume that S is a mob.*

(*) Assume that S is *compact*. Let E be the set of idempotents of S and let K be the minimal ideal of S [4]. We know that each $e\in E$ is contained in a maximal subgroup $H(e)$ of S and that no two such maximal subgroups intersect [8, Theorem 1]. Moreover ([1] and [8, Theorem 3]),

$$K = \bigcup\{H(e) \mid e \in E\cap K\},$$

and, if $e\in E$, then $e\in K$ if and only if $H(e)=eSe$. Finally, $K=\bigcup\{L\mid L\in\mathcal{L}\}$ where $\mathcal{L}$ is the set of minimal left ideals of S and $L\in\mathcal{L}$ if and only if $L=Sx$ for some $x\in K$ [1].

We refer to the above paragraph as (*). Its content is essential for the proofs of most of the theorems below. In some cases we may cite (*) and use instead the left-right dual of an assertion in (*).

Our purpose in this note is to examine the position of C-sets in clans relative to K and to the sets $H(e)$, $e\in E$. We show, for example, that if C is a C-set meeting $H(e)\neq\{e\}$, then $C\subset H(e)$. As noted earlier, topological groups may contain C-sets and it is easy to give examples of clans which are not groups and which contain C-sets. The results here are more decisive than those in [6].

The following lemma may be presumed to lie in the public domain, and we are unable to give any references.

LEMMA 1. *Let X be a continuum and let C be a C-set of X. Then C is connected and contains no inner points.*

The proof of the lemma consists of repeated applications of the familiar result that, if X is a continuum and if U is an open subset with $U^*\backslash U\neq\square$, then each component C of U satisfies $C^*\cap(U^*\backslash U)\neq\square$.

Received by the editors September 26, 1954.

[1] This work was done under Contract N7-onr-434, Task Order III, Navy Department, Office of Naval Research.

If X is a space and if $\{A_\lambda | \lambda \in \Lambda\}$ is a family of subsets of X indexed by a directed set Λ, then we define two subsets of X, sup A_λ and inf A_λ, as follows: $x \in \sup A_\lambda$ $\{x \in \inf A_\lambda\}$ if, for each open set U about x, there exists a cofinal $\{$residual$\}$ subset $\Lambda(U) \subset \Lambda$ such that $U \cap A_\lambda \neq \square$ for each $\lambda \in \Lambda(U)$. We write $\lim A_\lambda = A$, or $A_\lambda \to A$ if $\inf A_\lambda = A = \sup A_\lambda$. If Λ is the set of positive integers, then this notion of convergence is the usual one, given for example in detail in Kuratowski [**3**]. We shall suppose that the reader can supply the simple unstated results that we need concerning this notion.

Lemma 2. *Let S be a compact mob, and let $\{A_\lambda | \lambda \in \Lambda\}$ and $\{B_\lambda | \lambda \in \Lambda\}$ be two families of subsets of S based on the same directed set Λ. If $A_\lambda \to A$ and $B_\lambda \to B$, then $A_\lambda \cdot B_\lambda \to A \cdot B$* [**5**].

As in [**2**, Lemma 2], we may prove

Lemma 3. *If S is a continuum with left unit and if L is a left ideal of S, then $K \cup L$ is connected.*

Theorem 1. *Let S be a continuum with left unit, let L be a closed left ideal of S, and let C be a C-set of S. If C meets $K \cup L$ then $C \subset K \cup L$ (cf. [**6**, Theorem 1]).*

Proof. By Lemma 3 we know that $K \cup L$ is a continuum. We may assume that $K \cup L$ is a proper subset of C. Let $p \in C \backslash (K \cup L)$ and let U be an open set about p such that U^* does not meet $K \cup L$. Let M be the union of all left ideals of S contained in $S \backslash U^*$. Then M is an open [**2**, Lemma 1] left ideal which is connected by Lemma 3. Now M^* is a continuum not containing p and intersecting C. Hence $M^* \subset C$, contrary to the fact that, by Lemma 1, C contains no inner points.

Theorem 2. *If S is a clan and if K is a C-set, then K is a maximal subgroup of S.*

Proof. If S is a group then $K = S$ and the result follows. If S is not a group then $S \backslash H(u)$ is an ideal [**2**, Theorem 4] so that u is not in K and $S \backslash K$ is not void. Recall that S is connected and that K has no inner points by Lemma 1. Let C be an idempotent in K (see (*)) and let $\{a_\lambda | \lambda \in \Lambda\}$ be a directed set of points of $S \backslash K$ with $a_\lambda \to e$. Now $a_\lambda \in a_\lambda S$ and $a_\lambda S$ meets K because any right ideal meets any ideal. Since K is a C-set and $a_\lambda \in S \backslash K$ we must have $K \subset a_\lambda S$ for each λ. Thus $K \subset \bigcap \{a_\lambda S | \lambda \in \Lambda\} \subset \lim a_\lambda S = eS$, by Lemma 2. Thus $K \subset eS$ and dually $K \subset Se$ so that $K \subset eSe = eS \cap Se$. Now $e \in K$ gives $eSe \subset K$ and $eSe = H(e)$ (see (*)) and thus $K = H(e)$.

This result may fail if S has only a left unit.

THEOREM 3. *If S is a clan, if $e \in E$, and if C is a C-set meeting $H(e) \neq \{e\}$, then $C \subset H(e)$.*

PROOF. Since $H(e) \subset eSe = eS \cap Se$ we know that C intersects both Se and eS.

(A) Let $C \cap K = \square$. Now eS is a continuum which intersects K so that, of the two possibilities $C \subset eS$ and $eS \subset C$, we must have the former. Similarly $C \subset Se$ so that $C \subset eSe$. Now eSe is a clan with unit e and $H(e)$ is the maximal subgroup of eSe containing e. By Theorem 2 of [6] and Lemma 1 it follows that $C \subset H(e)$.

(B) Let $C \cap K \neq \square$ so that $C \subset K$ by Theorem 1. Thus $H(e)$ meets K and so $H(e) \subset K$ since $H(e)$ is a subgroup of S. Since $H(e) \subset eS \cap Se$ both eS and Se intersect C. There are four possibilities which we state and consider.

(i) $eS \cup Se \subset C$. We know (see (*)) that each $L \in \mathcal{L}$ meets eS. For any such $L \neq Se$ we have $L \subset C$ since L is a continuum ($L = Sx$ for some $x \in K$, see (*)) and $C \subset L$ would give $L \cap Se \neq \square$, contrary to the fact that L and Se (see (*)) are minimal left ideals. Since K is the union of all the minimal left ideals of S (see (*)) we know that $K \subset C$ and thus $C \subset K$ by Theorem 1. By Theorem 2, K is a maximal subgroup of S, i.e., $K = H(e)$, so that $C \subset H(e)$.

(ii) $eS \subset C \subset Se$. With the notation of (*) each $L \in \mathcal{L}$ intersects eS and thus Se and hence $L = Se$. It follows by (*) that $K = Se$. By the dual of Theorem 1 of [7], and that part of the proof of Theorem 4 of [7] on the middle of page 53, we know that there is a topological isomorphism $\phi: K \to H(e) \times (E \cap K)$. If $E \cap K$ is degenerate then $K = H(e)$ and $C \subset H(e)$. If $E \cap K$ contains more than one element then ($H(e) \neq \{e\}$ by assumption) the cartesian product of two nondegenerate continua contains a C-set, $\phi(C)$. Now a somewhat dull argument shows that $\phi(C)$ is a point and thus C is a point and hence $C \subset H(e)$.

(iii) $Se \subset C \subset eS$. This is the dual of (ii).

(iv) $C \subset Se$ and $C \subset eS$. Then $C \subset eS \cap Se = eSe = H(e)$ since $e \in K$ and using (*).

Although our last theorem does not involve C-sets, it fits easily into the pattern of this note.

With the aid of Theorem 4 and Lemma 3 of [2] as well as the argument used in the proof of these results we can prove

LEMMA 4. *If S is a clan with unit u and if $H(u)$ contains an inner point, then $H(u) = S$.*

THEOREM 4. *If S is a continuum, if $e \in K$, and if $H(e)$ has an inner point, then $H(e) = K$.*

PROOF. It is clear that eSe is a clan with unit e, that $H(e)$ is the maximal subgroup of eSe containing its unit and that $H(e)$ has inner points relative to eSe. Thus $H(e)=eSe$ by Lemma 4 and by (*) $H(e)\subset K$. Suppose that $H(e)\neq K$. By Lemma 3, K is connected so that there is a directed set of points $\{x_\lambda | \lambda\in\Lambda\}$ of $K\backslash H(e)$ with $x_\lambda\rightarrow x\in H(e)$. It follows from Lemma 2 that $x_\lambda Sx_\lambda\rightarrow xSx=eSe=H(e)$. Since $H(e)$ contains an inner point we know by definition that $H(e)\cap x_\lambda Sx_\lambda\neq\square$ for some $\lambda\in\Lambda$. Now $x_\lambda\in H(f)$ for some $f\in E\cap K$ (see (*)) and $H(f)=fSf$ readily gives $H(f)=x_\lambda Sx_\lambda$. Thus $H(e)$ meets $H(f)$ and, by (*), $H(e)=H(f)$ contrary to the fact that $x_\lambda\in S\backslash H(e)$ and $x_\lambda\in H(f)$.

REMARK. With the aid of Lemma 2 and (*) it easily follows that $\mathcal{L}$ is a continuous decomposition of K in this sense: If $\{x_\lambda | \lambda\in\Lambda\}$ is in K, if $x_\lambda\rightarrow x$, and if L_λ is the member of $\mathcal{L}$ containing x_λ then $L_\lambda\rightarrow L$, the member of $\mathcal{L}$ containing x. Thus if ϕ: $K\rightarrow\mathcal{L}$ is the natural function and if we give $\mathcal{L}$ the usual topology then ϕ is a continuous open function and $\mathcal{L}$ is a compact Hausdorff space. Of course we are supposing that S is compact. A similar result is obtained using the sets $H(e)$, $e\in E\cap K$.

REFERENCES

1. A. H. Clifford, *Semigroups containing minimal ideals*, Amer. J. Math. vol. 70 (1945) pp. 521–526.

2. R. J. Koch and A. D. Wallace, *Maximal ideals in compact semigroups*, Duke Math. J. vol. 21 (1954) pp. 681–686.

3. C. Kuratowski, *Topologie* I and II, Warsaw, 1948 and 1950.

4. K. Numakura, *On bicompact semigroups*, Math. J. Okayama University vol. 1 (1952) pp. 99–108.

5. W. L. Strother, *Continuity for multi-valued functions*, Tulane University dissertation, 1952.

6. A. D. Wallace, *Indecomposable semigroups*, Math. J. Okayama University vol. 3 (1953) pp. 1–3.

7. ———, *Cohomology, dimension and mobs*, Summa Brasiliensis Mathematicae vol. 3 (1953) pp. 43–54.

8. ———, *A note on mobs* II, Anais da Academia Brasileira de Ciencias vol. 25 (1953) pp. 335–336.

THE TULANE UNIVERSITY OF LOUISIANA