NOTE ON SUBDIRECT SUMS OF RINGS

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1. **Introduction.** If R is a given ring and $a \in R$, let us set

$$F(a) = \{ax - x;\ x \in R\},$$
$$G(a) = \{ax - x + \sum (y_i a z_i - y_i z_i);\ x,\ y_i,\ z_i \in R\},$$

the indicated sums being finite. Thus $F(a)$ is a right ideal of R, and $G(a)$ is the two-sided ideal of R generated by the elements of $F(a)$. Now let

$$J = \{a \in R;\ at \in F(at) \text{ for every } t \in R\},$$
$$N = \{a \in R;\ b \in G(b) \text{ for each element } b \text{ of } (a)\},$$

where (a) denotes the two-sided ideal of R generated by a. Then J is the *Jacobson radical* [6] of R, and N is the *radical* of R as defined in [1]. It is well known that $J=0$ if and only if R is isomorphic to a subdirect sum of primitive rings, and $N=0$ if and only if R is isomorphic to a subdirect sum of simple rings with unit element.

The above definitions of J and N suggest that it might be of some interest to consider the set

$$M = \{a \in R;\ at \in G(at) \text{ for every } t \in R\},$$

which is obtained by replacing $F(at)$ by $G(at)$ in the definition of J. It is then natural to inquire whether anything can be said about subdirect sum representations of R in relation to the condition that $M=0$. The purpose of this note is to give a brief discussion of this problem. Before proceeding, we make a few preliminary remarks about the set M.

By their definitions, it is clear that $J \subseteq N \subseteq M$, and that these all coincide if the ring R is commutative.

Next we observe that M contains every nilpotent element of R. For if $z^n=0$, then

$$zt = -\sum_{i=1}^{n-1} [z^i(zt)t^i - z^i t^i],$$

and thus $zt \in G(zt)$ for every element t of R. It follows that *if $M=0$, then R can contain no nonzero nilpotent elements.*

Now it is known that J and N are (two-sided) ideals of R. However,

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in general M need not be an ideal. As an example, let R be the ring of all matrices of order n over a field. Since R is a simple ring and $G(at)$ is an ideal, it follows that $G(at)$ is either 0 or the entire ring R. Now $G(at)=0$ if and only if at is a left unit of the simple ring R, therefore the unit element of R. In other words, $a \in M$ if and only if a has no inverse. Hence M consists of the singular matrices, and M is therefore not an ideal in R.

2. **Necessary and sufficient condition that** $M=0$. For convenience of reference, we shall say that a subdirect sum *has property* $\mathcal{P}$ if each of the component rings is a simple ring with unit element, and at least one component of *every* nonzero element has a right inverse. A ring R has property $\mathcal{P}$ if it is isomorphic to a subdirect sum with property $\mathcal{P}$.

We shall now prove the following theorem.

THEOREM 1. *A necessary and sufficient condition that the ring R have property $\mathcal{P}$ is that $M=0$.*

First, assume that R has property $\mathcal{P}$ and let us identify R with the subdirect sum with property $\mathcal{P}$ to which it is isomorphic. If a is any nonzero element of R, then some component of a, say a_α, has a right inverse y_α in the component ring R_α. Let y be an element of R such that the α-component of y is y_α. It follows that $ay \notin G(ay)$ since otherwise we would have $a_\alpha y_\alpha \in G(a_\alpha y_\alpha)=0$, which is impossible since $a_\alpha y_\alpha$ is the unit element of R_α. It follows that $a \notin M$, and hence $M=0$.

Conversely, assume that $M=0$. If $a \neq 0$, there is an element t of R such that $at \notin G(at)$. By Zorn's Lemma, we can extend $G(at)$ to an ideal K_a which is maximal in the set of those ideals not containing a, and clearly K_a is maximal in R. Now for every element y of R,

$$aty - y \equiv 0 \ (K_a),$$

that is, at is a left unit element modulo K_a. Since R/K_a is simple, at is the unit element of R modulo K_a and t is a right inverse of a modulo K_a. Since for each nonzero element a of R there exists such an ideal K_a, clearly $\bigcap_{a \in R} K_a=0$, and the subdirect sum of the rings R/K_a to which R is isomorphic clearly has the property $\mathcal{P}$. This completes the proof of the theorem.

3. **Subdirect sums with property** $\mathcal{P}$. In this section we shall indicate a few results about subdirect sums with property $\mathcal{P}$. First we prove the following theorem.

THEOREM 2. *Let S be a subdirect sum of rings $S_\alpha (\alpha \in \mathfrak{A})$ which has* *property* $\mathfrak{P}$. *If a and b are elements of S such that* $ab=0$, *then for every* $\alpha\in\mathfrak{A}$, *we have* $a_\alpha=0$ *or* $b_\alpha=0$.

Let a and b be elements of S such that $ab=0$. Clearly S has no nonzero nilpotent elements, and thus $ab=0$ implies that $(ba)^2=0$ and hence that $ba=0$. Thus the set I_a of all elements x of S such that $ax=0$ is an ideal in S, and $(b)\subseteq I_a$. Let b_α be any fixed nonzero component of b, and therefore b_α is a nonzero element of the simple ring S_α with unit element 1_α. It follows that $(b_\alpha)=S_\alpha$, and hence for suitably chosen elements $x_\alpha^{(i)}$, $y_\alpha^{(i)}$ of S_α, we have

$$1_\alpha = \sum x_\alpha^{(i)} b_\alpha y_\alpha^{(i)}.$$

If $x^{(i)}$, $y^{(i)}$ are elements of S with respective α-components $x_\alpha^{(i)}$, $y_\alpha^{(i)}$, then $\sum x^{(i)}by^{(i)}\subseteq(b)$, and hence $a(\sum x^{(i)}by^{(i)})=0$. Taking α-components, we find that

$$a_\alpha(\sum x_\alpha^{(i)} b_\alpha y_\alpha^{(i)}) = a_\alpha 1_\alpha = a_\alpha = 0.$$

Next we shall prove the following result.

THEOREM 3. *Let a be a nonzero element of a subdirect sum S of rings* S_α $(\alpha\in\mathfrak{A})$ *which has property* $\mathfrak{P}$. *If a is regular, every nonzero component of a has an inverse; if a is not regular, an infinite number of components of a have right inverses.*

Suppose first that a is regular, that is, that there exists an element y of R such that $aya=a$. Since S has no nonzero nilpotents, the idempotent ay is in the center of S [3]. Hence, for each α, $a_\alpha y_\alpha$ is an idempotent in the center of the simple ring S_α and hence is the zero or the unit element of S_α. However, $a_\alpha\neq 0$ implies that $a_\alpha y_\alpha\neq 0$ since $a_\alpha=a_\alpha y_\alpha a_\alpha$, and thus every nonzero component a_α has a right inverse y_α. Similar arguments show that y_α also is a left inverse and hence the inverse of a_α.

Now assume that a is not regular. Since $a\neq 0$, some component, say a_α, of a has a right inverse x_α in S_α. Let x be an element of S with x_α as α-component. Then $b=axa-a$ is not zero since a is not regular, and $b_\alpha=0$. Moreover, some component b_β of b has a right inverse y_β. Thus $(a_\beta x_\beta a_\beta - a_\beta)y_\beta = 1_\beta$, or

$$a_\beta(x_\beta a_\beta y_\beta - y_\beta) = 1_\beta,$$

and a_β has a right inverse in S_β. Now let y be an element of S with β-component y_β, and consider the element $c=byb-b$. Now c is not regular (in particular, not zero) since it is easy to see [2, Lemma 1] that this would imply that b is regular, and hence also that a is regular. Hence some nonzero component c_γ of c has a right inverse.

Since $c_\alpha = c_\beta = 0$, we know that γ is distinct from α and β, and it follows by arguments similar to those used above that b_γ has a right inverse, and finally that a_γ has a right inverse. Evidently this process can be continued to establish the theorem.

From this result, together with Theorem 1, we obtain immediately the following corollary.

COROLLARY. *If R is a regular ring, $M = 0$ if and only if R is isomorphic to a subdirect sum of division rings.*

Since it is known [3] that a regular ring is isomorphic to a subdirect sum of division rings if and only if it has no nonzero nilpotent elements, we see that for a regular ring R, $M = 0$ if and only if R has no nonzero nilpotent elements.

Now let $(a)_r$ denote the principal right ideal generated by a. Then, in the notation of the proof of Theorem 3, we have the properly descending chain

$$(a)_r \supset (b)_r \supset (c)_r \supset \cdots .$$

Hence if $M = 0$ and the descending chain condition holds for principal right ideals, R must be a regular ring and, by the preceding corollary, is isomorphic to a subdirect sum of division rings. An argument of Kaplansky [5, p. 74] shows that this subdirect sum is in fact a discrete direct sum of division rings. It follows therefore that *if $M = 0$ and the descending chain condition holds for principal right ideals of R, then R is a discrete direct sum of division rings.* Now Gertschikoff [4] has shown that the same conclusion follows if the condition that $M = 0$ be replaced by the condition that R have no nonzero nilpotent elements. Hence these two conditions are equivalent in the presence of the descending chain condition for principal right ideals.

REFERENCES

1. B. Brown and N. H. McCoy, *Radicals and subdirect sums*, Amer. J. Math. vol. 69 (1947) pp. 46–58.

2. ———, *The maximal regular ideal of a ring*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 165–171.

3. A. Forsythe and N. H. McCoy, *On the commutativity of certain rings*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 523-526.

4. A. Gertschikoff, *Über Ringe, die in eine direkte Summe von Körpern zerlegbar sind*, Mat. Sbornik N. S. vol. 7 (1940) pp. 591–597 (Russian with German summary).

5. I. Kaplansky, *Topological representation of algebras*, Trans. Amer. Math. Soc. vol. 68 (1950) pp. 62–75.

6. N. Jacobson, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math. vol. 67 (1945) pp. 300–320.

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