THE SYLOW SUBGROUPS OF THE SYMMETRIC GROUPS

A. J. WEIR

The aim of this paper is to give a direct approach to the study of the Sylow p-subgroups S_n of the symmetric group of degree p^n. [We assume throughout that $p \neq 2$.] Many of the results are already known and are treated in a paper by Kaloujnine where he uses a particular representation by means of "reduced polynomials."[1] It has seemed worth while to restate some of his results using the concept of *complete product* $L \circ M$ of two permutation groups L, M which he and Krasner have recently emphasised.[2] This elementary notion is of great importance in the theory of finite groups and it appears in the literature in different forms.[3] A simple interpretation is given in §2 in terms of permutation matrices which shows the strong connection between the operation $\circ$ and imprimitivity. The associative rule for $\circ$ follows from the associative law of matrix multiplication and we show that $S_n \cong C \circ C \circ \cdots \circ C$ (n factors) where C is cyclic of order p; various expressions of S_{m+n} as $S_m \circ S_n$ allow us to express S_{m+n} as a split extension and to investigate many properties of S_{m+n} by an inductive process. Since the Sylow p-subgroups of the classical groups (general linear, symplectic, orthogonal and unitary) over a finite field with characteristic prime to p are direct products of the basic subgroups $\overline{S}_n \cong \overline{C} \circ C \circ \cdots \circ C$ (n factors) where $\overline{C}$ is cyclic of order p^r ($r \geqq 1$),[4] it is hoped that the treatment here may also suggest ways in which these groups $\overline{S}_n$ may be studied.

$S_{n+1} \cong C \circ S_n$ and so $S_{n+1} \cong A^n \cdot S_n$ where A^n is elementary abelian of order p^{p^n}. We show that all the factors of the series $A^n > (A^n, S_n) > (A^n, S_n, S_n) > \cdots$ are cyclic of order p and this leads naturally to a description of certain subgroups of S_n in terms of partition diagrams. The main result of the paper is to show finally that the characteristic subgroups are precisely the normal partition subgroups.[5]

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[1] See [2] and [3, part I, p. 212, note 5].

[2] See [3] and [4] (summary of results contained in [3]).

[3] [5, §172] gives an inductive construction of S_n which is essentially the same as ours. The complete product is the Kranz group of Pólya [6, p. 178]. See also [7].

[4] [9].

[5] There is a strong similarity between the partition subgroups treated here and the partition subgroups in [8].

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1. **Retractions and lower central series.** A *retraction* ϕ of a group G is a homomorphism of G into itself for which $\phi^2=\phi$.

ϕ has kernel H and image L, say. Clearly ϕ acts as the identity on L.

I. $H\cap L=1$, for if $x\in H\cap L$, then $\phi(x)=1$ and $\phi(x)=x$.

II. $HL=G$. If $g\in G$ put $\phi(g)=x$ and $g=yx$. Then $x=\phi(y)x$, so that $\phi(y)=1$ and $y\in H$.

Conversely if G is a *split extension* $G=HL$, $H\cap L=1$, H normal in G, then each element of G is uniquely expressible in the form yx ($y\in H$, $x\in L$) and $\phi: yx\rightarrow x$ is a retraction.

There is a (1–1) *correspondence between retractions of G and expressions of G as a split extension.*

Under a retraction ϕ the lower central series $G_1\supset G_2\supset\cdots$ of G maps onto the lower central series $L_1\supset L_2\supset\cdots$ of L and $\phi_k=\phi|G_k$ is clearly a retraction of G_k. If H_k is defined inductively: $H_1=H$, and $H_{k+1}=(H_k, G)$, we shall show that H_k is the kernel of ϕ_k and hence

THEOREM 1. $G_k=H_kL_k$ $(k=1, 2, \cdots)$.

Suppose H_r is the kernel of ϕ_r then the kernel K of ϕ_{r+1} certainly contains (H_r, G) and (G_r, H), since $\phi(H)=1$. Now $\phi(y_rx_r, yx)=(x_r, x)$ and so by expanding the commutator we see that K is in any normal subgroup of G containing (H_r, G) and (G_r, H). The result now follows by the

LEMMA 1. *If P is a normal subgroup of G and we define by induction $P_1=P$, $P_{r+1}=(P_r, G)$, then $(G_k, P)\subset(P_k, G)$.*

The lemma is true for $k=1$; assume $(G_{r-1}, P)\subset P_r$ for all P. Now $(G_r, P)=(G_{r-1}, G, P)$. Also $(G, P, G_{r-1})=(P_2, G_{r-1})\subset P_{r+1}$ by the induction hypothesis applied to P_2, and $(P, G_{r-1}, G)\subset(P_r, G)=P_{r+1}$ by the induction hypothesis. Hence $(G_r, P)\subset P_{r+1}$.[6]

REMARK. If H is abelian we may simplify (H_k, G) to the form (H_k, L).

2. **Complete products.** If G is a permutation group of degree r we apply $\sigma\in G$ to the columns of 1_r and obtain a matrix $M(\sigma)$. Postmultiplication of an $(r\times r)$ matrix by $M(\tau)$ permutes the columns by

[6] If A, B, C are any subgroups of G, then any normal subgroup which contains (A, B, C) and (B, C, A) also contains (C, A, B). See [10, p. 47] or [11, Chap. II, §6, Theorem 14].

τ and so in particular $M(\sigma)M(\tau)=M(\sigma\tau)$. It follows that $\sigma\rightarrow M(\sigma)$ is a faithful representation of G. Premultiplication of an $(r\times r)$ matrix by $M(\sigma^{-1})$ applies a permutation to the rows which must be σ since $M(\sigma^{-1})M(\sigma)=1$.

Let G be imprimitive with systems of imprimitivity $1, \cdots, m$; $m+1, \cdots, 2m; \cdots; \cdots, mn$. Any $\sigma\in G$ induces a permutation σ' of these systems. In matrix notation σ' becomes an $(n\times n)$ matrix $M^*(\sigma')$ of $(m\times m)$ blocks obtained by applying σ' to the "columns" of 1_{mn}. [This is just the usual $M(\sigma')$ magnified m times.] If $\sigma'=1$, σ permutes the systems of imprimitivity inside themselves and so may be written $\sigma=\tau_1\cdots\tau_n$ where the τ_i commute among themselves. These considerations suggest the following definition:

Given permutation groups A, B of degrees m, n respectively, the group of all

$$\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{bmatrix} B^*$$

where $A_i\in M(A)$ and $B^*\in M^*(B)$ is denoted by $A\circ B$ and is called the *complete product* of A and B.

$A\circ B$ has degree mn, and may again be viewed as a permutation group. If B^* represents the permutation $b\in B$, transformation by B^* applies b to the rows and columns and so

$$B^{*-1}\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{bmatrix} B^* = \begin{bmatrix} A_{b(1)} & & \\ & \ddots & \\ & & A_{b(n)} \end{bmatrix}.$$

If $\mathfrak{S}_m$ is the symmetric group of degree m then the above construction shows that G is an imprimitive permutation group if and only if G is a subgroup of some $\mathfrak{S}_m\circ\mathfrak{S}_n$.

If $G=A\circ B$ the homomorphism $\phi:\sigma\rightarrow\sigma'$ is clearly a retraction of G onto B with kernel $(A)^n$ (direct product), and so G is expressed as a split extension

$$(A)^n\cdot B, \qquad (A)^n\cap B = 1.$$

Since matrix multiplication is associative it follows that $A\circ(B\circ C)\cong(A\circ B)\circ C$ for any three permutation groups A, B, C. We may thus write down without ambiguity such expressions as $S_n\cong C\circ C\circ\cdots\circ C$ (n factors) where C is cyclic of order p. This S_n is a permutation group of degree p^n and order $p^{1+p+\cdots+p^{n-1}}$ and so is a Sylow p-subgroup of $\mathfrak{S}_{p^n}$. A Sylow p-subgroup of any $\mathfrak{S}_r$ is a direct product of such basic[7] S_n so we shall restrict our discussion to them.

3. **The special case S_2.** S_2 may be expressed in terms of generators $x; z_1, \cdots, z_p$ each of order p, such that the z's commute and $x^{-1}z_ix = z_{i+1}$ (suffixes mod p).

A is the elementary abelian subgroup generated by $z_1, \cdots, z_p$. We define a new basis of A as follows:

$$y_0 = z_1, \qquad (y_i, x) = y_{i+1} \qquad (i = 0, 1, \cdots).$$

Then $y_0 = z_1$, $y_1 = z_2z_1^{-1}, \cdots, y_k = z_{k+1}z_k^{-k} \cdots z_1^{(-1)^k}$ and so in particular $y_{p-1} = z_1 \cdots z_p$ and $y_p = 1$ $(p \neq 2)$.

We may take $y_0, \cdots, y_{p-1}$ as basis of A. The chain $A > (A, S_1) > \cdots$ written $A_0 > A_1 > \cdots$ is given by the rule $A_i = [y_i, y_{i+1}, \cdots]$.

We notice the important fact that each factor has order p. The lower central series is $S_2 > A_1 > A_2 > \cdots > A_p = 1$ so S_2 has class p.

The upper and lower central series coincide.[8]

If $a \in A$ we define $a_0 = a$, $(a_i, x) = a_{i+1}$ and then

$$(xa)^2 = x^2a^xa \qquad = x^2a^2a_1$$
$$(xa)^k = x^ka^{x^{k-1}} \cdots a^xa = x^ka^ka_1^{C_{k,2}} \cdots a_{k-1}$$

and so

$$(xa)^p = a^{1+x+\cdots+x^{k-1}} \qquad = a_{p-1}.$$

If $a = z_1^{\sigma_1} \cdots z_p^{\sigma_p}$, $(xa)^p = (z_1 \cdots z_p)^\sigma$ where $\sigma = \sum_1^p \sigma_i$. The group generated by the pth powers of the elements of S_2 is A_{p-1}, the centre. Also $(xa)^p = 1$ if and only if $a \in A_1$.

The proper normal subgroups of S_2 are just the A_i and the subgroups of index p. Of the latter A and $B = [x, y_1, \cdots, y_{p-1}]$ are characteristic. A is the unique maximal abelian normal subgroup of S_2 and B is the only other normal subgroup of S_2 of index p whose elements are all of order p or 1.

4. **The partition subgroups of S_n.** The natural retraction of S_{n+1} onto S_n gives the split extension $S_{n+1} \cong A^nS_n$ where A^n is elementary abelian of order p^{p^n}.

THEOREM 2. *The factors of $A^n > (A^n, S_n) > (A^n, S_n, S_n) > \cdots$ are all of order p.*[9]

We write the above series as $A_0^n > A_1^n > \cdots$. As a Sylow p-subgroup

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[7] Introduction to [1] and [2].

[8] See [5, §172, Ex. 2].

[9] This corresponds to Lemmas 1 and 2 in [2].

of $\mathfrak{S}_{p^n}$, S_n clearly contains a cycle x of order p^n.[10] By suitable numbering we may take $z_1, \cdots, z_{p^n}$ as generators of A^n and $x^{-1}z_ix=z_{i+1}$ (suffixes mod p^n). Define $y_0=z_1$, $(y_i, x)=y_{i+1}$. As before, $y_{p^n-1}=z_1z_2 \cdots z_{p^n}$, $y_{p^n}=1$. A fortiori $A^n_{p^n-1}>1$ and the theorem follows.

This result shows that any normal subgroup H of S_{n+1} contained in A^n is of the form A^n_i. [$y_i \in H$ implies $y_{i+1} \in H$.]

REMARK. $S_{n+1} \cong (S_n)^p \cdot C$ and so A^{n-1} in S_n corresponds to A^n in S_{n+1}. Also A^{n-1}_1 in S_n corresponds to a normal subgroup of S_{n+1} of index p^p in A^n, viz. A^n_p. (Normal, since C merely permutes the direct factors.) Hence C transforms A^n mod A^n_p in the same way as A^1 (in S_2).

The product of a subgroup and a normal subgroup is again a subgroup, so $P=A^0_{i_0}A^1_{i_1} \cdots A^n_{i_n}$ is a subgroup of S_{n+1}. We may draw a diagram with $n+1$ columns with p^j squares in the $j+1$th column and then P is represented by a partition $|P|$ of this diagram. We call P a *partition subgroup*.

THEOREM 3. *The lower central series $H_0>H_1> \cdots$ of S_n is obtained by removing successive rows from the top of the partition diagram.*

This follows immediately from Theorems 1 and 2 by induction.

We put $T_m=A^mA^{m+1} \cdots A^n$ and say that an element or subgroup has *depth m* if it is contained in T_m but not in T_{m+1}.

LEMMA 2. *If x has depth m, the least normal subgroup of S_{n+1} containing (A^n, x) is $A^n_{p^m}$.*[11]

$S_{n+1} \cong T_mS_m$; $T_m \cong (S_k)^{p^m}$ $(k=n-m+1)$. Now (A^n, x) in T_m corresponds to $(A^{n-m}, x_1) \times \cdots \times (A^{n-m}, x_{p^m})$ where for some j, $x_j \in S_k$ has depth 0. The lemma is true for $m=0$ by Theorem 2 so this direct product contains $1 \times \cdots \times 1 \times A^{n-m}_1 \times 1 \cdots \times 1$. We may permute the factors by transforming by the cycle in S_m, so any normal subgroup of S_{n+1} containing (A^n, x) must contain $A^n_{p^m}$.

REMARK. (If x has depth m) we may apply the same argument with A^{n-m}_1 of S_k instead of A^{n-m} and see that the least normal subgroup of S_{n+1} containing $(A^n_{p^m}, x)$ is $A^n_{2p^m}$.

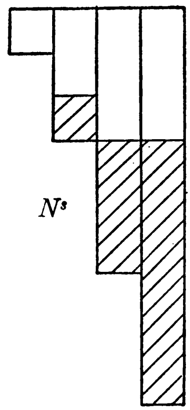
THEOREM 4. *A necessary and sufficient condition for a partition subgroup P of depth s to be normal is $i_r \leqq p^s$ (all r).*

We write $P=HL$, $H \subset A^n$, $L \subset S_n$. Then $(P, S_{n+1}) \subset P$ if and only if $(L, S_n) \subset L$ and $(L, A^n) \subset H$. This last condition is equivalent to $i_n \leqq p^s$ by Lemma 2 so the theorem is obvious by induction.

[10] [5, §172, Ex. 1].

[11] This corresponds to Lemmas 3 and 4 of [2].

If N_i^s is the least normal subgroup of S_{n+1} containing A_i^s $(i < p^s)$ then $N_i^s \cap S_{k+1}$ is normal in S_{k+1} and so by Lemma 2 must contain $A_{p^s}^k$ $(k \leqq n)$. By the above theorem $A_i^s \theta_{s+1}$ is a normal subgroup so $N_i^s = A_i^s \theta_{s+1}$. Here we write θ_{s+1} for H_{p^s} (and in fact $\theta_0 > \theta_1 \cdots$ is the derived series).[12]



Given two *distinct* squares (i, j), (u, v) in $|S_n|$; we shall say (i, j) *covers* (u, v) if $(u, v) \in |N_i^j|$. When $|P|$ is a normal partition we shall say $|P|$ *covers* (u, v) if some square of $|P|$ does. If (u, v) covers some square outside $|P|$ we shall say (u, v) *avoids* $|P|$.

When P is a normal partition subgroup of S_n we define $P' = (P, S_n)$ and P^* where $P^*/P =$ centre of S_n/P. Then P' and P^* are again (normal) partition subgroups:

THEOREM 5. (i) $|P'|$ *consists of the squares covered by* $|P|$.
(ii) $|P^*|$ *consists of the squares which do not avoid* $|P|$.

(i) If $P = HL$, $H \subset A^r$, $L \subset S_r$, P' is the least normal subgroup of S_{r+1} containing (H, S_r), (L, A^r), and L'. The first two are precisely the parts of A^r covered by H and L respectively so (i) follows by induction on r.

(ii) Let $|\overline{P}|$ be the set of squares which do not avoid $|P|$. By (i), $(\overline{P}, S_n) \subset P$. If $x \notin \overline{P}$ then any normal subgroup of S_n containing x must contain some A_i^j where (i, j) avoids $|P|$. Hence $\overline{P}$ is the greatest normal subgroup of S_n for which $(\overline{P}, S_n) \subset P$.

As an immediate application we have:

COROLLARY. *The upper and lower central series of S_n coincide.*

5. **The characteristic subgroups of S_n.** Transforming S_n by ele-

[12] See [2].

ments of the normalizer of S_n in $\mathfrak{S}_{p^n}$ produces certain *diagonal* automorphisms as well as the inner automorphisms. These do not give the whole automorphism group but we shall see that they are sufficient to "cut out" all the characteristic subgroups. The "diagonal" nature of these automorphisms is perhaps most clearly seen in the Kaloujnine tableau notation.[13] We shall merely quote the elementary fact that there is such an automorphism $uv \to u^2 v$ of S_r ($p \neq 2$) where $u \in A^r$, $v \in S_r$.

LEMMA 3. *Every characteristic subgroup of S_n is a normal partition subgroup.*

[We shall prove by induction that any subgroup invariant under inner and diagonal automorphisms is a normal partition subgroup.]

Suppose K is a characteristic subgroup of S_{r+1}, $L = K \cap S_r$, $H = K \cap A^r$. L is invariant under inner and diagonal automorphisms of S_r and so is a normal partition subgroup of S_r by the induction hypothesis.

H is normal in S_{r+1} and $H \subset A^r$, so $H = A_i^r$ for some i. If $uv \in K$, where $u \in A^r$, $v \in S_r$, then $u^2 v \in K$ (by the above remark) and so $u, v \in K$. Finally $K = HL$.

THEOREM 6. *A^n is the unique maximal abelian normal subgroup of S_{n+1}.*

Put $H = A^n_{p^n-1}$ and G = centralizer of H. A^n is abelian and so $A^n \subset G$. If $x \notin A^n$, x has depth $\leqq n-1$ and so $(H, x) \not\subset A^n_{2p^{n-1}+1}$ (Lemma 2, Remark). But $p^n > 2p^{n-1}+1$ ($p \neq 2$), and so $x \notin G$. Hence $A^n = G$.

If B is any abelian normal subgroup of S_{n+1}, then $B \cap A^n = A_i^n$ for some i. If B contains x of depth $\leqq n-1$, then B is a normal subgroup of S_{n+1} containing (A^n, x) and so $B \supset H$. But B is abelian and so $B \subset A^n$, contradiction. Hence in any case $B \subset A^n$.

By this result we see that there is a dual of the derived series and hence $A^n, A^n A^{n-1}, \cdots$ are characteristic in S_{n+1}. This shows incidentally that the depth of an element is invariant under automorphisms.

We know for S_2 that the elements of order p and depth 0 all lie in $A^0\theta_1$. Suppose we have proved this for S_n. Let xy be an element of order p and depth 0 in S_{n+1} where $x \in S_n$, $y \in A^n$. Clearly $x^p = 1$ (working mod A^n), and so $x \in A^0\theta_1$ in S_n. Also C transforms A^n/A^n_p and A^1 in the same way (Remark, Theorem 2). Thus $(xy)^p \equiv y_{p-1}$ (mod A^n_p) and $y \in A^n_1$, i.e. $xy \in A^0\theta_1$ in S_{n+1}.

[13] See [2].

Suppose now that x is an element of order p and depth m in S_n, then x corresponds to an element $(x_1, x_2, \cdots, x_{p^m})$ of $S_{n-m}^{p^m}$ of order p. If each x_i has depth 0 in S_{n-m} it follows that $x_i \in A^0\theta_1$ and $x \in A^m\theta_{m+1}$.

In S_2, x has depth 0 if and only if the group of elements of A^1 which commute with x has order p.

$(S_2)^{p^m} \cong T_m/T_{m+2}$. Hence $x_1, \cdots, x_{p^m}$ all have depth 0 if and only if the group of elements of T_{m+1} which commute with x mod T_{m+2} has order p^m. This condition is invariant under automorphisms.

If $z_1, \cdots, z_{p^m}$ are the generators of A^m, the least normal subgroup of S_{m+1} containing z_1 also contains the other z's and so is A^m itself. $z_1 z_2 \cdots z_{p^m} \in A^m_{p^m-1}$. The least normal subgroup of S_n containing $x = z_1^2 z_2 \cdots z_{p^m}$ is therefore $A^m\theta_{m+1}$. Since $p \neq 2$, the "parts" of x are all of depth 0. Hence under an automorphism $A^m\theta_{m+1}$ maps into itself, i.e. $A^m\theta_{m+1}$ is characteristic.

All normal partition subgroups are obtained by commuting these $A^m\theta_{m+1}$ with S_n and taking products. Hence in conjunction with Lemma 3 we have finally:

THEOREM 7. *The characteristic subgroups of S_n are precisely the normal partition subgroups.*

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PRINCETON UNIVERSITY