A NOTE ON BERNSTEIN'S APPROXIMATION PROBLEM

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This note is intended as an addendum to the paper [2] with which we presuppose familiarity.

Vidav [3] has observed that if condition (3) of [2] is satisfied:

(3) *there exists a sequence of polynomials* p_n *such that*

$$\lim_{n\to\infty} p_n(u)K(u) = 1, \qquad |p_n(u)K(u)| \leqq C, \qquad -\infty < u < \infty,$$

then $\{u^n K(u)\}_0^\infty$ *is fundamental in* $C_0(-\infty, \infty)$, *unless* $1/K(z)$ *is an entire function.*

Unfortunately this criterion is defective in that it leaves in an inconclusive position such functions as $K(u) = e^{-u^2}$, $K(u) = \text{sech } u$ whose reciprocals are entire, which satisfy the condition (3), and yet which are known to give rise to fundamental sets $\{u^n K(u)\}$.

In this note we present a sharper criterion which leaves in this indecisive state a smaller class of functions.

THEOREM. *If* (3) *holds, then* $\{u^n K(u)\}$ *is fundamental in* $C_0(-\infty, \infty)$ *unless* $1/K(z)$ *is an entire function of exponential type zero.*

PROOF. If (3) holds and $\{u^n K(u)\}$ is not fundamental, then necessarily [2]

$$\int_{-\infty}^{\infty} \frac{\log |K(u)|}{1+u^2} du > -\infty.$$

Let $F = 1/K$. By virtue of (3) K does not vanish so that F is continuous and

$$\int_{-\infty}^{\infty} \frac{\log |F(u)|}{1+u^2} du < \infty.$$

Since K vanishes at $\pm\infty$, $|F(u)| > 1$ for sufficiently large u. Therefore the preceding formula is equivalent to

$$\text{(A)} \qquad \int_{-\infty}^{\infty} \frac{\log^+ |F(u)|}{1+u^2} du < \infty.$$

According to (3) and the principle of harmonic majorization [1, p. 955]

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$$
\begin{aligned}
\log |p_n(z)| &\leqq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y| \log |p_n(\xi)|}{(x-\xi)^2+y^2}\, d\xi \\
\text{(B)} \qquad &\leqq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y| \log^+ |F(\xi)|}{(x-\xi)^2+y^2}\, d\xi + \log C \\
&\equiv U(x, y), \; y \neq 0, \; z = x + iy.
\end{aligned}
$$

The function $U(x, y)$ is harmonic in $y>0$ and in $y<0$. As $x+iy\to x_0$, $U(x, y)\to U(x_0, 0)$, by a well-known property of the Poisson integral of a continuous function. Hence $|p_n(z)|$ is majorized by the continuous function $\exp\,(U(x, y))$. Since $p_n(x)\to 1/K(x)$ on the real axis, it follows that $\lim_{n\to\infty} p_n(z)$ exists for all z and is an entire function, $F(z)$. On the real axis $F(x)=1/K(x)$, of course. Letting $n\to\infty$ in (B) gives

$$
\text{(C)} \qquad \log |F(z)| \leqq U(x, y).
$$

In $|y| \geqq \max\,(|x|, 2)$,

$$
\frac{1+\xi^2}{(x-\xi)^2+y^2} < A.
$$

Also in $\xi^2<|z|$, $|z|>2$

$$
\frac{1+\xi^2}{(x-\xi)^2+y^2} < \frac{B}{|z|}\cdot
$$

Hence in $|y| \geqq \max\,(|x|, 2)$,

$$
\begin{aligned}
U(x, y) &= \frac{|y|}{\pi}\int_{-\infty}^{\infty} \frac{\log^+ |F(\xi)|}{1+\xi^2}\,\frac{1+\xi^2}{(x-\xi)^2+y^2}\, d\xi \\
&= \frac{|y|}{\pi}\int_{|\xi|^2<|z|} + \frac{|y|}{\pi}\int_{\xi^2\geqq |z|} \\
&< O\left(\frac{|y|}{|z|}\int_{\xi^2<|z|} \frac{\log^+ |F(\xi)|}{1+\xi^2}\, d\xi\right) \\
&\quad + O\left(|y| \int_{\xi^2>|z|} \frac{\log^+ |F(\xi)|}{1+\xi^2}\, d\xi\right) = o(|y|),
\end{aligned}
$$

since the integral in the second O-term is $o(1)$, by (A). This proves by (C) that $F(z)$ is of order one, minimum type in $\pi/4 \leqq |\theta| \leqq 3\pi/4$. To complete the proof note that

$$|p_n(\pm re^{\pm i\pi/4})| \leqq e^{\epsilon r} \qquad (r > r_0(\epsilon)).$$

We can now apply a standard Phragmén-Lindelöf argument.

The function $f(z) = p_n(z)e^{-2^{1/2}\epsilon z}$ satisfies

$$|f(z)| < K(\epsilon) \qquad (|\theta| \leqq \pi/4, |z| = r_0(\epsilon)),$$

$$|f(z)| < 1 \qquad (|\theta| = \pi/4; |z| > r_0(\epsilon); |\theta| \leqq \pi/4, |z| = R),$$

where R is any sufficiently large positive number. Hence, by the maximum modulus principle

$$|f(z)| < K(\epsilon),$$

i.e.

$$|p_n(z)| < K(\epsilon)|e^{2^{1/2}\epsilon z}| < K(\epsilon)e^{\epsilon|z|}(|\theta| \leqq \pi/4, |z| \geqq r_0(\epsilon)).$$

It follows that there is a function $r_1(\eta)$ of η such that for every $\eta > 0$

$$|p_n(z)| < e^{\eta|z|} \qquad (|\theta| \leqq \pi/4, |z| \geqq r_1(\eta)).$$

Letting $n \to \infty$ gives

$$|F(z)| \leqq e^{\eta|z|} \qquad (|\theta| \leqq \pi/4, |z| \geqq r_1(\eta)).$$

Similarly this inequality can be proved for the sector $|\pi - \theta| \leqq \pi/4$, which completes the proof of the theorem.

REFERENCES

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