SOME APPLICATIONS OF THEOREMS ON UNIFORM CAUCHY POINTS TO INFINITE SERIES[1]

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1. Introduction. Uniform Cauchy points and points of uniform convergence of a sequence of functions are important in certain convergence questions.[2] They will be applied here to obtain results on series in pseudo-metric semigroups by topologizing $P(N)$, the permutation group of the natural numbers N, and setting up the sequence of partial sum functions of the series as did Agnew [**1**] and Sengupta [**8**].

It has long been known that $P(N)$ is a metric space with the Fréchet metric defined by

$$d_F(t, t') = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{| t(i) - t'(i) |}{1 + | t(i) - t'(i) |}$$

for each t and t' in $P(N)$, and that if $\{t_n\}$ is a sequence in $P(N)$ and $t \in P(N)$, then $\lim_{n\to\infty} d_F(t_n, t) = 0$ if and only if $\lim_{n\to\infty} t_n(i) = t(i)$ for each $i \in N$. If X is a set and d is a pseudo-metric for X let $(X, \mathfrak{T}(d))$ denote the topological space consisting of the set X and topology $\mathfrak{T}(d)$ where $\mathfrak{T}(d)$ has the basis $[V(x, \epsilon): x \in X, \epsilon > 0]$ where $V(x, \epsilon) = [x': d(x, x') < \epsilon]$. Agnew [**1**] shows that $P(N)$ with the metric d_F is not complete. Nevertheless he sketches a proof of the fact that $(P(N), \mathfrak{T}(d_F))$ is a Baire space, i.e., each of its non-null open sets is second category. Banach [**2**, p. 229] noted that $P(N)$ with the metric d_B defined by

$$d_B(s, t) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{| s(i) - t(i) | + | s^{-1}(i) - t^{-1}(i) |}{1 + | s(i) - t(i) | + | s^{-1}(i) - t^{-1}(i) |}$$

for each s and t in $P(N)$ is a complete metric space and if one uses composition of functions as multiplication, $(P(N), \mathfrak{T}(d_B))$ is a topo-

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[2] See, for example, [**4**; **5**; **6**]. In this connection the theorem of a recent note of A. Huber, *Note on a theorem of Ostrowski*, which appeared in these Proceedings, June, 1954, is a special case of [**4**, Corollary 3.1].

logical group. Schreier and Ulam [7] have noted that $P(N)$ with the metric $d_{F'}$ defined by $d_{F'}(s, t)=d_F(s, t)+d_F(s^{-1}, t^{-1})$ for each s and t in $P(N)$ is a complete metric space. One may easily verify that $d_F(s, t)\leqq d_B(s, t)\leqq d_{F'}(s, t)$ for all s and t in $P(N)$. Also it is known that if $\{s_n\}$ is a sequence in $P(N)$ and $s\in P(N)$, then $\lim_{n\to\infty} d_F(s_n, s)=0$ if and only if $\lim_{n\to\infty} d_F(s_n^{-1}, s^{-1})=0$. It follows from these remarks and the definition of $d_{F'}$ that if $\{s_n\}$ is a sequence in $P(N)$ and $s\in P(N)$, then $\lim_{n\to\infty} s_n=s$ with respect to one of the metrics d_F, d_B, and $d_{F'}$ if and only if $\lim_{n\to\infty} s_n=s$ with respect to all of them. Therefore all three metrics induce the same topology on $P(N)$, i.e., $\mathfrak{T}(d_F)=\mathfrak{T}(d_B)=\mathfrak{T}(d_{F'})$. Since $P(N)$ with the metric d_B is a complete metric space it follows that $(P(N), \mathfrak{T}(d_B))$ is a Baire space and hence $(P(N), \mathfrak{T}(d_F))$ is a Baire space since $\mathfrak{T}(d_F)=\mathfrak{T}(d_B)$. This way of showing that $(P(N), \mathfrak{T}(d_F))$ is a Baire space differs from the way outlined by Agnew [1].

Still another metric, d_A, may be defined for $P(N)$ by setting $d_A(s, t)=0$ if s and t are the same permutation of N and if s and t are distinct permutations of N by setting $d_A(s, t)=1/n$ where n is the least natural number i such that $s(i)\neq t(i)$. By a remark of Agnew [1], $\mathfrak{T}(d_F)=\mathfrak{T}(d_A)$. Because of its simplicity and its equivalence to d_F, the metric d_A will be used in the sequel to verify statements about $(P(N), \mathfrak{T}(d_F))$.

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2. **Lemmas for the main theorem.** Suppose Y is a topological space with a uniform topology, as defined in [9], determined by the uniformity $[V_\alpha: \alpha\in A]$. Suppose further that $\Lambda=[\lambda, \geqq]$ is a directed set and X is a topological space and that $\{f_\lambda\}$ is a family of functions such that for each $\lambda\in\Lambda$, f_λ is a function on X to Y. An element x of X is a *point of convergence* of $\{f_\lambda\}$ if $\{f_\lambda(x)\}$ is a convergent net in Y. The set of points of convergence of $\{f_\lambda\}$ will be denoted by $C'(f_\lambda)$. The defining property in the next definition was used by Osgood [5, p. 161]. A point $x_0\in X$ is a *point of uniform convergence* of $\{f_\lambda\}$ if there is $y\in Y$ such that for each $\alpha\in A$ there is an open set G_α containing x_0 and there is $\lambda_\alpha\in\Lambda$ with $f_\lambda(x)\in V_\alpha(y)$ for all $x\in G_\alpha$ and $\lambda\geqq\lambda_\alpha$. The set of points of uniform convergence of $\{f_\lambda\}$ will be denoted by $C_u'(f_\lambda)$. A point x_0 in X is a *Cauchy point* of $\{f_\lambda\}$ [6] if and only if $\{f_\lambda(x_0)\}$ is a Cauchy net in Y. The set of Cauchy points of $\{f_\lambda\}$ will be denoted by $C(f_\lambda)$. A point x_0 in X is a *uniform Cauchy point of* $\{f_\lambda\}$ [6] if and only if to each $\alpha\in A$ there corresponds $\lambda_\alpha\in\Lambda$ and an open set G containing x_0 such that $f_\lambda(x)\in V_\alpha(f_{\lambda_\alpha}(x))$ for all $\lambda\geqq\lambda_\alpha$ and all x in G.

LEMMA 1. *The relations* $C'(f_\lambda) \subset C(f_\lambda)$ *and* $C'_u(f_\lambda) \subset C_u(f_\lambda)$ *always hold. If* $C'(f_\lambda) = C(f_\lambda)$ *and each* f_λ *is continuous then* $C'_u(f_\lambda) = C_u(f_\lambda)$.

PROOF. The first statement of the theorem is evident. Because of it the second statement will follow if one shows that $C_u(f_\lambda) \subset C'_u(f_\lambda)$. Assume the hypothesis of the second statement in the theorem and suppose $x_0 \in C_u(f_\lambda)$. Let α be an element of A. Then there is $\beta \in A$ such that if p, q, and r are elements of Y such that $p \in V_\beta(r)$ and $q \in V_\beta(r)$, then $p \in V_\alpha(q)$. Since $x_0 \in C_u(f_\lambda)$ it follows from [6, Lemma 1] that $x_0 \in C(f_\lambda) \cap C_a(f_\lambda)$ where $C_a(f_\lambda)$ is the set of points of almost equicontinuity of $\{f_\lambda\}$. By definition $x_0 \in C_a(f_\lambda)$ means that for arbitrary $\beta \in A$ (and hence for the β of the preceding sentence) there is $\lambda_\beta \in \Lambda$ and an open set $G \ni x_0$ such that $f_\lambda(x) \in V_\beta(f_\lambda(x_0))$ for all $\lambda \geqq \lambda_\beta$ and $x \in G$. Since $x_0 \in C(f_\lambda) = C'(f_\lambda)$ there is $y_0 \in Y$ such that the net $\{f_\lambda(x_0)\}$ is convergent to y_0. Next there is $\gamma \in A$ such that if s, t, and u are elements of Y and $s \in V_\gamma(u)$ and $t \in V_\gamma(u)$, then $s \in V_\beta(t)$; also by symmetry of this condition $t \in V_\beta(s)$. Because $\{f_\lambda(x_0)\}$ is convergent to y_0 there is $\lambda_\gamma \in \Lambda$ such that if $\lambda \geqq \lambda_\gamma$ then $f_\lambda(x_0) \in V_\gamma(y_0)$. Choose some $\lambda_\alpha \in \Lambda$ such that $\lambda_\alpha \geqq \lambda_\beta$ and $\lambda_\alpha \geqq \lambda_\gamma$ and suppose $\lambda \in \Lambda$ and $\lambda \geqq \lambda_\alpha$. Then one has that $f_\lambda(x_0) \in V_\gamma(y_0)$ and $y_0 \in V_\gamma(y_0)$ so $f_\lambda(x_0) \in V_\beta(y_0)$ and $y_0 \in V_\beta(f_\lambda(x_0))$. Now if x is an element of G it follows that $f_\lambda(x) \in V_\beta(f_\lambda(x_0))$ since $\lambda \geqq \lambda_\beta$. Hence, $f_\lambda(x) \in V_\beta(f_\lambda(x_0))$ and $y_0 \in V_\beta(f_\lambda(x_0))$ so $f_\lambda(x) \in V_\alpha(y_0)$. It has been shown that $x_0 \in C'_u(f_\lambda)$. The next lemma is a special case of part of [4, Theorem 3].

LEMMA 2. *Suppose* X *is a topological space and* Y *is a pseudo-metric space and* $\{f_n\}$ *is a sequence of continuous functions on* X *to* Y. *If* $C(f_n)$ *is second category so is* $C_u(f_n)$.

Throughout the remainder of this paper $\mathcal{J}$ will denote the family of non-null, finite subsets of the natural numbers N. Also let $[1, n] = [k \in N: 1 \leqq k \leqq n]$; and for any set S let $\emptyset$ stand for the void subset of S. The topology $\mathfrak{T}(d_P)$ will be used for $P(N)$ throughout the paper.

LEMMA 3. *If* X *is an additively written semigroup with a topology and* $s = \{s(i)\}$ *is a sequence in* X, *then for each* $n \in N$ *the function* $\sigma[n, s]$ *defined on* $P(N)$ *to* X *by* $\sigma[n, s](t) = \sum_{i=1}^{n} s(t(i))$ *for each* $t \in P(N)$ *is continuous.*

PROOF. Suppose $t \in P(N)$ and H is an open subset of X containing $\sigma[n, s](t)$. If $t' \in P(N)$ and $d_A(t, t') < 1/n+1$, then $t(i) = t'(i)$ for $i \in [1, n]$ so $\sigma[n, s](t') = \sigma[n, s](t) \in H$. Hence, $\sigma[n, s]$ is continuous.

The following well known fact about permutation is also needed.

LEMMA 4. *Suppose* $t_0 \in P(N)$, $n \in N$, *and* $D = \{d_j\}_{j=1}^p$ *is a finite set of* $p > 0$ *positive integers and* $t_0([1, n]) \cap D = \emptyset$. *Then there is an* $n' > n$ *and* $t \in P(N)$ *such that* $t(i) = t_0(i)$ *for all* $i \in [1, n]$; $t(n+j) = d_j$ *for all* $j \in [1, p]$; *and* $t(i) = t_0(i)$ *for* $i > n'$.

3. The main theorem.

THEOREM 1. *Suppose* $s = \{s(i)\}$ *is a sequence in a commutative semigroup* X *with a topology, and suppose that* ψ *is a real valued continuous function on* X *such that*:

(1) $\psi(x+y) \leqq \psi(x) + \psi(y)$ *for all* x *and* y *in* X.

(2) $\psi(y) \leqq \psi(x+y) + |\psi(x)|$ *for all* x *and* y *in* X.

Let

$$A = \left[t \in P(N) : \sup \left\{ \psi\left(\sum_{i \in D} s(t(i)) \right) : D \in \mathcal{J} \right\} < +\infty \right]$$

and let

$$B = \left[t \in P(N) : \sup \left\{ \psi\left(\sum_{i=1}^{n} s(t(i)) \right) : n \in N \right\} < +\infty \right].$$

Then the following conclusions are valid:

(i) *The set* B *is an* F_σ *subset of* $(P(N), \mathfrak{T}(d_F))$.

(ii) *If* A *is non-null then* $A = B = P(N)$ *and there is a real number* K *such that* sup $[\psi(\sum_{i \in D} s(t(i))) : D \in \mathcal{J}$ *and* $t \in P(N)] \leqq K$.

(iii) *If* A *is null then* B *is first category in* $(P(N), \mathfrak{T}(d_F))$ *and* $P(N) \setminus B$ *is a second category everywhere dense* G_δ *subset of* $P(N)$.

(iv) *If* B *is a second category subset of* $(P(N), \mathfrak{T}(d_F))$ *then* $A = B = P(N)$.

(v) *The set* $P(N) \setminus B$ *is residual in* $(P(N), \mathfrak{T}(d_F))$ *if and only if it is non-null.*

PROOF. For each $n \in N$ define a function F_n on $P(N)$ into the reals by $F_n(t) = \max\ [\psi(\sum_{i=1}^{j} s(t(i)) : j \in [1, n]]$ for each $t \in P(N)$. By Lemma 3, for each $j \in N$ the function S_j defined on $P(N)$ to X by $S_j(t) = \sum_{i=1}^{j} s(t(i))$ for each $t \in P(N)$ is continuous. Then ψS_j is continuous on $P(N)$ to the reals for each $j \in N$ since it is the composition of continuous functions. Therefore since $F_n(t) = \max\ [\psi(S_j(t)) : j \in [1, n]]$ for each $t \in P(N)$, F_n is continuous for each $n \in N$. Now for each $t \in P(N)$ the sequence $\{F_n(t)\}$ is a nondecreasing sequence of real numbers. Hence, $C'(F_n)$, the convergence set of the nondecreasing sequence of continuous functions $\{F_n\}$, is an F_σ subset of $P(N)$.

Furthermore, it is clear that $t \in B$ if and only if $\lim_{n\to\infty} F_n(t)$ exists and is finite, i.e., $B = C'(F_n)$. This proves that B is an F_σ subset of $P(N)$ and hence that $P(N)\backslash B$ is a G_δ subset of $P(N)$.

It is obvious that $A \subset B$. Suppose A is non-null. Then there is $t_0 \in P(N)$ and a finite real number K such that

$$\sup\left[\psi\left(\sum_{i\in D} s(t_0(i))\right): D \in \mathcal{J}\right] \leqq K.$$

Let t and D be arbitrary elements of $P(N)$ and $\mathcal{J}$ respectively. Since t and t_0 are elements of $P(N)$ there is $D_0 \in \mathcal{J}$ such that $t(D) = t_0(D_0)$. Therefore $\psi(\sum_{i\in D} s(t(i))) = \psi(\sum_{i\in D_0} s(t_0(i))) \leqq K$. This completes the proof of (ii).

Statement (iii) will now be verified. Suppose $B = C'(F_n)$ is a second category subset of $P(N)$. By Lemma 1 and the completeness of the reals it follows that $C'(F_n) = C(F_n)$ and $C'_u(F_n) = C_u(F_n)$. Since $C'(F_n) = C(F_n)$ is second category in $P(N)$ and $\{F_n\}$ is a sequence of continuous functions it follows by Lemma 2 that $C_u(F_n) = C'_u(F_n)$ is also second category in $P(N)$. Hence there exists $t_0 \in C'_u(F_n)$. Therefore $\epsilon = 1$ implies there is $n_0 \in N$ such that if $t \in G \equiv [t' \in P(N): d_A(t_0, t') < 1/n_0]$ and $n \geqq n_0$ then $F_n(t) < (\lim_{n\to\infty} F_n(t_0)) + 1$. Because of the definition of d_A this can be rephrased as follows:

If $t \in P(N)$ and $t(i) = t_0(i)$ for $i \in [1, n_0]$ then

$$(*) \qquad \psi\left(\sum_{i=1}^{j} s(t(i))\right) < \lim_{n\to\infty} F_n(t_0) + 1 \qquad \text{for every } j \text{ in } N.$$

It will be shown by considering four cases that the identity of $P(N)$ is an element of A. Let D be an arbitrary but fixed element of $\mathcal{J}$. Let $K \equiv \max\left(\sum_{i=1}^{n_0} |\psi(s(t_0(i)))|, \lim_{n\to\infty} F_n(t_0) + 1\right)$. In case

$$D \subset t_0([1, n]),$$

using property (1) of ψ one has

$$\psi\left(\sum_{i\in D} s(i)\right) \leqq \sum_{i\in D} \psi(s(i)) \leqq \sum_{i\in D} |\psi(s(i)))| \leqq \sum_{i=1}^{n_0} |\psi(s(t_0(i)))| \leqq K.$$

Now consider the case where $t_0([1, n_0])$ is a proper subset of D. Let $\{d_j\}_{j=1}^{p}$ denote the elements of $D\backslash t_0([1, n_0])$. Then by Lemma 4 there is $t \in p(N)$ such that $t(i) = t_0(i)$ for $i \in [1, n_0]$ and $t(n_0 + j) = d_j$ for $j \in [1, p]$. Using property (*) it follows that

$$\psi\left(\sum_{i\in D} s(i)\right) = \psi\left(\sum_{i=1}^{n_0+p} s(t(i))\right) < K$$

which finishes the verification of this case. Next suppose that $t_0([1, n_0])\cap D$ is the null set. In this case, using property (2) of ψ, it follows that

$$\psi\left(\sum_{i\in D} s(i)\right) \leqq \psi\left(\left(\sum_{i=1}^{n_0} s(t_0(i))\right) + \sum_{i\in D} s(i)\right) + \left|\psi\left(\sum_{i=1}^{n_0} s(t_0(i))\right)\right|.$$

Let $d_1, d_2, \cdots, d_p$ denote any ordering of the elements of D. By Lemma 2 there is $t\in P(N)$ such that $t(i)=t_0(i)$ for $i\in[1, n_0]$ and $t(n_0+j)=d_j$ for $j\in[1, p]$. For such a permutation t the equality $(\sum_{i=1}^{n_0} s(t_0(i)))+\sum_{i\in D} s(i)=\sum_{i=1}^{n_0+p} s(t(i))$ holds. From this together with proposition (*) it follows that the first term on the righthand side of the preceding inequality is no greater than K. Using property (1) of ψ the second term is no greater than K so $\psi(\sum_{i\in D} s(i))\leqq 2K$ in this case. Finally suppose both

$$t_0([1, n_0]) \cap D \text{ and } F \equiv D\backslash t_0([1, n_0])$$

are non-null. Let $E\equiv t_0([1, n_0])\backslash D$. Set $y=\sum_{i\in D} s(i)$ and $x=\sum_{i\in E} s(i)$. By property (2) of ψ one then has $\psi(y)\leqq\psi(x+y)+|\psi(x)|$. Note that $x+y=\sum_{i=1}^{n_0} s(t_0(i))+\sum_{i\in F} s(i)$. Using Lemma 4 there is $t_1\in P(N)$ and $n\in N$ such that $t_1(i)=t_0(i)$ for $i\in[1, n_0]$ and $\sum_{i=1}^{n} s(t_1(i))=x+y$. Therefore by proposition (*), $\psi(x+y)\leqq K$. That $|\psi(x)|\leqq K$ follows from the inequality $|\psi(x)|=|\psi(\sum_{i\in E} s(i))|\leqq\sum_{i=1}^{n_0}|\psi(s(t_0(i)))|$ and the definition of K. Hence, in this case also $\psi(y)\leqq 2K$.

To verify statement (iv) suppose that B is second category. Then by (iii), A is non-null so by (ii), $A=B=P(N)$.

To verify statement (v) suppose that $P(N)\backslash B$ is residual in $P(N)$. Since $P(N)$ is second category in itself, $P(N)\backslash B$ is also second category in $P(N)$ so it is non-null. Conversely, if $P(N)\backslash B$ is non-null then B is first category in $P(N)$, i.e., $P(N)\backslash B$ is residual. If B were not first category then B would equal $P(N)$ by (iv) which contradicts the supposition that $P(N)\backslash B$ is non-null.

4. **Applications.** N. Bourbaki [**3**] has shown that conditions (iii) and (v) in the following theorem are equivalent if X is any abelian topological group.

THEOREM 2. *Suppose* $(X, +, \mathfrak{T}(d))$ *is a complete commutative semigroup with a topology* $\mathfrak{T}(d)$ *determined by the pseudo metric* d. *Let* $s=\{s(i)\}$ *denote a sequence in* X *and for each* $n\in N$ *let* $\sigma(n, s)$ *denote* *the function on $P(N)$ to X defined in Lemma 3. The following conditions on s are pairwise equivalent.*

(i) $C'(\sigma[n, s]) \equiv [t \in P(N)$: *the sequence* $\{\sum_{i=1}^{n} s(t(i))\}_{n=1}^{\infty}$ *is convergent*] *is second category in* $(P(N), \mathfrak{T}(d_F))$.

(ii) $C'_u(\sigma[n, s]) \neq \emptyset$.

(iii) *There is an* $x \in X$ *such that to each* $\epsilon > 0$ *there corresponds* $F_\epsilon \in \mathcal{J}$ *such that if* $F \in \mathcal{J}$ *and* $F \supset F_\epsilon$, *then* $d(\sum_{i \in F} s(i), x) < \epsilon$.

(iv) $C'_u(\sigma[n, s]) = P(N)$.

(v) $C'(\sigma[n, s]) = P(N)$, *i.e., the series* $\sum_{i=1}^{\infty} s(i)$ *is unconditionally convergent.*

PROOF. Since $P(N)$ is second category, (v) obviously implies (i).

Next assume that $C'(\sigma[n, s])$ is second category. Since X is complete, $C'(\sigma[n, s]) = C(\sigma[n, s])$ so, by Lemma 1, $C'_u(\sigma[n, s]) = C_u(\sigma[n, s])$. Since $C'(\sigma[n, s])$ is second category in $P(N)$ it follows using Lemma 2 that $C'_u(\sigma[n, s])$ is second category in $P(N)$ which proves that (i) implies (ii) since any second category set is non-null.

Suppose that $C'_u(\sigma[n, s]) \neq \emptyset$. Therefore there is $t_0 \in P(N)$ with these properties:

(*) There exists $x \in X$ such that to each $\epsilon > 0$ there corresponds $n_\epsilon \in N$ such that if $t \in P(N)$ and $t(i) = t_0(i)$ for $i \in [1, n_\epsilon]$, then $d(\sum_{i=1}^{n} s(t(i)), x) < \epsilon$ for all $n \geqq n_\epsilon$.

To show that property (*) implies (iii) let $\epsilon > 0$ be given. Let n_ϵ be the natural number whose existence is assured by (*) and let $F_\epsilon = [t_0(i): i \in [1, n_\epsilon]]$. Suppose $F \in \mathcal{J}$ and $F \supset F_\epsilon$. Let $n_\epsilon + p$ denote the number of elements in F. By Lemma 4 there is $t \in P(N)$ such that $t(i) = t_0(i)$ for $i \in [1, n_\epsilon]$ and $t([1, n_\epsilon + p]) = F$. Using (*) it follows that $d(\sum_{i \in F} s(i), x) = d(\sum_{i=1}^{n_\epsilon + p} s(t(i)), x) < \epsilon$. This proves that (ii) implies (iii).

Next suppose that (iii) holds, that $t_0 \in P(N)$, and that $\epsilon > 0$. Let F_ϵ denote an element of $\mathcal{J}$ satisfying statement (iii). Choose n_ϵ large enough so that $t_0([1, n_\epsilon]) \supset F_\epsilon$. If $t \in P(N)$ and $t(i) = t_0(i)$ for $i \in [1, n_\epsilon]$ and $n \geqq n_\epsilon$, then $t([1, n]) \supset F_\epsilon$ so $d(\sum_{i=1}^{n} s(t(i)), x) < \epsilon$. This proves that $t_0 \in C'_u(\sigma[n, s])$ and hence that (iii) implies (iv).

Obviously (iv) implies (v). This completes the proof of Theorem 2.

THEOREM 3. *Suppose* $(X, +, \mathfrak{T}(d))$ *is a commutative semigroup with a zero* θ *and a topology* $\mathfrak{T}(d)$ *determined by a pseudo metric* d *such that* $d(x, y) = d(z + x, z + y)$ *for all* x, y, *and* z *in* X. *If* $s = \{s(i)\}$ *is a sequence in* X *such that* $[t \in P(N)$: $\sup \{d(\sum_{i=1}^{n} s(t(i)), \theta): n \in N\} < \infty]$ *is a second category subset of* $(P(N), \mathfrak{T}(d_F))$, *then there is a real number* K *such that* $\sup [d(\sum_{i \in D} s(t(i)), \theta): D \in \mathcal{J}, t \in P(N)] \leqq K$.

PROOF. Define a function ψ on X into the reals by $\psi(x) = d(x, \theta)$ for each $x \in X$. All hypotheses of Theorem 1 are satisfied as can easily be verified, and hence from (iv) and (ii) of that theorem the conclusion follows.

COROLLARY 3.1. *If X is a weakly complete Banach space and $\sum_{i=1}^{\infty} x(i)$ is a series whose terms are elements of X, then $\sum_{i=1}^{\infty} x(i)$ is unconditionally convergent if and only if*

$$B \equiv \left[t \in P(N) : \sup \left\{ \left\| \sum_{i=1}^{n} x(t(i)) \right\| : n \in N \right\} < \infty \right]$$

is second category in $(P(N), \mathfrak{T}(d_F))$.

PROOF. If $\sum_{i=1}^{\infty} x(i)$ is unconditionally convergent, then for each $t \in P(N)$ the sequence $\{ \sum_{i=1}^{n} x(t(i)) \}_{n=1}^{\infty}$ is a Cauchy sequence so it is bounded in X. Hence $B = P(N)$ which is second category. Conversely suppose B is second category in $P(N)$. Using the usual metric $d(x, y) = \|x - y\|$ for each x and y in X it is immediate from the conclusion of Theorem 3, taking t to be the identity permutation, that $\sup [\| \sum_{i \in F} x(i) \| : F \in \mathcal{J}] < \infty$ and it is well known that this condition implies that $\sum_{i=1}^{\infty} x(i)$ converges unconditionally since X is a weakly complete Banach space.

Agnew [1] noted that if $\sum_{i=1}^{\infty} z(i)$ is a convergent series of complex numbers for which $\sum_{i=1}^{\infty} \|z(i)\| = +\infty$ then the set of $t \in P(N)$ for which $\sum_{i=1}^{\infty} z(t(i))$ has bounded partial sums is a set of the first category. The following result which includes his is obtained by a different proof.

COROLLARY 3.2. *If $s = \{s(i)\}$ is a sequence in a Banach space X with finite dimension and $\sum_{i=1}^{\infty} \|s(i)\| = +\infty$ then*

$$E \equiv \left[t \in P(N) : \sup \left\{ \left\| \sum_{i=1}^{n} s(t(i)) \right\| : n \in N \right\} = +\infty \right]$$

is a residual everywhere dense G_δ subset of $(P(N), \mathfrak{T}(d_F))$.

PROOF. Let $\psi(x) = \|x\|$. By (i) of Theorem 1, E is a G_δ subset of $P(N)$. Suppose E is not residual, i.e., suppose $P(N) \backslash E$ is second category. Then using Theorem 3 there exists a finite number K such that $\sup [\| \sum_{i \in F} s(i) \| : F \in \mathcal{J}] \leqq K$. It is well known that, for a Banach space with finite dimension, the above boundedness condition implies that the series $\sum_{i=1}^{\infty} s(i)$ converges unconditionally. This however is a contradiction because unconditional convergence implies absolute convergence when X has finite dimension. Therefore E is residual and hence since it is a residual G_δ subset of a Baire space it is everywhere dense.

If $s=\{s(i)\}$ is a sequence of real numbers let $E_{+\infty}(s)=[t\in P(N)$: $\limsup_n \sum_{i=1}^n s(t(i))=+\infty]$ and let

$$E_{-\infty}(s) = \left[t \in P(N) \colon \liminf_n \sum_{i=1}^n s(t(i)) = -\infty \right].$$

Agnew [1] and Sengupta [8] have proved the following theorem:

THEOREM 4. *If $s=\{s(i)\}$ is a sequence of real numbers such that $\lim \sum_{i=1}^n s(i)$ exists and $\sum_{i=1}^\infty |s(i)|=+\infty$, then $E_{+\infty}(s)\cap E_{-\infty}(s)$ is a residual everywhere dense G_δ subset of $(P(N), \mathfrak{T}(d_F))$.*

The hypothesis of Theorem 4 is sufficient but not necessary for its conclusion. For example it follows from Theorem 6 that if s is defined at each $i\in N$ by $s(i)=1^{i-1}$, then the conclusion of Theorem 4 holds for s. However, $\sum_{i=1}^\infty s(i)$ is not a convergent series. Part (iii) of Theorem 5 gives one necessary and sufficient condition for the conclusion of Theorem 4 and Theorem 6 gives another.

THEOREM 5. *The following statements are valid for any sequence $s=\{s(i)\}$ of real numbers.*

(i) *The set $E_{+\infty}(s)$ is a residual everywhere dense G_δ subset of $(P(N), \mathfrak{T}(d_F))$ if and only if $E_{+\infty}(s)$ is non-null.*

(ii) *The set $E_{-\infty}(s)$ is a residual everywhere dense G_δ subset of $P(N)$ if and only if $E_{-\infty}(s)$ is non-null.*

(iii) *The set $E_{+\infty}(s)\cap E_{-\infty}(s)$ is a residual everywhere dense G_δ subset of $P(N)$ if and only if $E_{+\infty}(s)$ is non-null and $E_{-\infty}(s)$ is non-null.*

PROOF. Let ψ denote the identity map of the additive group of the real numbers with its usual topology. Note that ψ satisfies the hypothesis of Theorem 1. Using B as defined in that theorem it is clear that $B=[t\in P(N)$: $\limsup_n \sum_{i=1}^n s(t(i))<\infty]$ so $E_{+\infty}(s)=P(N)\backslash B$. By (i) of Theorem 1, $E_{+\infty}(s)$ is a G_δ subset of $P(N)$ and by (v) of Theorem 1, $E_{+\infty}(s)$ is residual in $P(N)$ if and only if it is non-null. Since $E_{+\infty}(s)$ is a G_δ subset of the Baire space $P(N)$ it is everywhere dense in $P(N)$ when and only when it is residual in $P(N)$. This proves statement (i). Statement (ii) follows from (i) and the fact that $E_{-\infty}(s)=E_{+\infty}(-s)$ where $-s$ is defined at each $i\in N$ by $-s(i)$. Since $P(N)$ is second category in itself, if $E_{+\infty}(s)\cap E_{-\infty}(s)$ is residual in $P(N)$ then it is non-null so $E_{+\infty}(s)$ and $E_{-\infty}(s)$ are non-null. Conversely if $E_{+\infty}(s)$ and $E_{-\infty}(s)$ are non-null then by (i) and (ii) each is a residual G_δ subset of $P(N)$ and therefore so is their intersection. Hence their intersection is everywhere dense since $P(N)$ is a Baire space.

For the sake of brevity the following fairly evident lemma is stated without proof.

LEMMA 5. *Suppose* $s=\{s(i)\}$ *is a sequence of real numbers. Then,* $E_{+\infty}(s)$ *is non-null if and only if* $\sum_{i=1}^{\infty} s(t_0(i))=+\infty$ *where* $\{s(t_0(i))\}$ *is the subsequence of non-negative terms of* s. *Also,* $E_{-\infty}(s)$ *is non-null if and only if* $\sum_{i=1}^{\infty} s(t_1(i))=-\infty$ *where* $\{s(t_1(i))\}$ *is the subsequence of negative terms of* s.

THEOREM 6. *If* $s=\{s(i)\}$ *is a sequence of real numbers, then* $E_{+\infty}(s) \cap E_{-\infty}(s)$ *is a residual everywhere dense* G_δ *subset of* $(P(N), \mathfrak{T}(d_F))$ *if and only if* $\sum_{i=1}^{\infty} s(t_0(i))=+\infty$ *and* $\sum_{i=1}^{\infty} s(t_1(i))=-\infty$ *where* $\{s(t_0(i))\}$ *and* $\{s(t_1(i))\}$ *are the subsequences of non-negative and negative terms of* s *respectively.*

PROOF. The conclusion follows immediately from (iii) of Theorem 5 together with Lemma 5.

If $s=\{s(i)\}$ is a sequence of real numbers such that $\lim_{n\to\infty}\sum_{i=1}^{n} s(i)$ exists and $\sum_{i=1}^{\infty}|s(i)|=+\infty$ then it follows, using the notation of Theorem 6, that $\sum_{i=1}^{\infty} s(t_0(i))=+\infty$ and $\sum_{i=1}^{\infty} s(t_1(i))=-\infty$. Hence, the Agnew-Sengupta theorem, Theorem 4, is a corollary of Theorem 6.

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