EXTENSIONS OF MAPPINGS INTO n-CUBES[1]

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1. Introduction. Throughout this paper X is a separable metric space. If n is a positive integer, I^n is the closed unit cube in Euclidean n-space. If A is a subset of X and f is a mapping whose domain contains A, then *f is of type k on A* if and only if dim $(A \cap f^{-1}(y)) \leqq k$ for each y in the range of f.

Let us assume that $m \geqq n$, dim $(X) = m$, A is a closed subset of X and ϕ is a mapping of A into I^n. It is shown that if ϕ is of type $m-n$ on A, then ϕ can be extended to a mapping f of X into I^n such that f is of type $m-n$ on X. An interesting special case of this result is the following theorem: If X has dimension n and ϕ is a light mapping of a closed subset A of X into I^n, then ϕ can be extended to a light mapping of X into I^n.

It is also shown that if X is compact and $m \geqq n$, then dim $(X) \leqq m$ if and only if there exists a mapping f on X into I^n such that f is of type $m-n$ on X. For the special case $m=n$, this result was obtained by M. L. Curtis and G. S. Young in [1].

2. An extension theorem. A finite collection Σ of subsets of a metric space has *property* D if and only if there exists $\eta > 0$ such that any set which contains at least one point from each member of Σ has diameter greater than η.

LEMMA 1. *If $A_0, \cdots, A_n$ are nonempty separable metric spaces of dimension less than n, f_j is a mapping of A_j into I^n, and $\epsilon > 0$, then for each $j = 0, \cdots, n$ there exists a mapping g_j of A_j into I^n such that $\|g_j(x) - f_j(x)\| < \epsilon$ for all $x \in A_j$ and the collection $g_0(A_0), \cdots, g_n(A_n)$ has property* D.

PROOF. Let N be a positive integer for which $1/N < \epsilon/(2n)$, and let $0 < \delta_0 < \delta_1 < \cdots < \delta_n < 1/N$ be $n+1$ positive real numbers. For each j, $0 \leqq j \leqq n$, we define S_j to be the set of all points in I^n at least one of whose coordinates is of the form $(k/N) + \delta_j$ for some integer k, $0 \leqq k \leqq N-1$. It is easily seen that $S_0, \cdots, S_n$ has property D.

For each j, $0 \leqq j \leqq n$, the components of $I^n - S_j$ are parallelepipeds of diameter less than $\epsilon/2$. Thus, for $0 \leqq j \leqq n$, there exists a finite sub-

set F_j of $I^n - S_j$ for which there is an $(\epsilon/2)$-retraction R_j of $I^n - F_j$ onto S_j.

Since the dimension of each space A_j is less than n, every value of the mapping f_j is unstable (see [3, p. 74]). It follows easily that there exists a mapping h_j of A_j into $I^n - F_j$ such that $\|h_j(x) - f_j(x)\| < \epsilon/2$ for each $x \in A_j$. We now define g_j to be the composite mapping $R_j h_j$. It is easy to see that $\|g_j(x) - f_j(x)\| < \epsilon$ for all $x \in A_j$, and since $g_j(A_j) \subset S_j$, it follows that $g_0(A_0), \cdots, g_n(A_n)$ has property D.

We let C_n be the space of mappings of X into I^n, metrized by the uniform metric. If K is a closed subset of X and h is a mapping into I^n whose domain contains K, we let $C_n(h|K)$ be the subset of C_n which consists of all $f \in C_n$ for which $f|K = h|K$. $C_n(h|K)$ is a closed subset of C_n, and hence is itself a complete metric space under the uniform metric. As a special case it is convenient to permit K to be the empty set $\varnothing$, in which case $C_n(h|\varnothing) = C_n$ for every h.

LEMMA 2. *If K is a closed subset of X, h is a mapping of K into I^n, and $A_0, \cdots, A_n$ are mutually exclusive subsets of $X - K$ which are closed and each of dimension less than n, then the set G of all mappings $f \in C_n(h|K)$ for which $f(A_0), \cdots, f(A_n)$ has property D is open and dense in $C_n(h|K)$.*

PROOF. Suppose $f \in G$. There exists $\eta > 0$ such that if $x_0, \cdots, x_n$ are points of $A_0, \cdots, A_n$ respectively, then the diameter of the set whose points are $f(x_0), \cdots, f(x_n)$ is greater than η. It follows that if $g \in C_n(h|K)$ and the distance from g to f is less than $\eta/3$, then the collection $g(A_0), \cdots, g(A_n)$ has property D and hence $g \in G$. This proves that G is open.

Now suppose that $f \in C_n(h|K)$ and $\epsilon > 0$. There exists an $(\epsilon/2)$-retract R of the $(\epsilon/2)$-neighborhood of I^n onto I^n. We define $f_j = f|A_j$ and use Lemma 1 to obtain mappings $g_0, \cdots, g_n$ of $A_0, \cdots, A_n$ respectively into I^n for which $\|g_j(x) - f_j(x)\| < \epsilon/2$ for $x \in A_j$ and for which the collection $g_0(A_0), \cdots, g_n(A_n)$ has property D. We define a mapping ϕ on $K \cup \bigcup_{j=0}^{n} A_j$ into N, the $(\epsilon/2)$-neighborhood of the origin θ of Euclidean n-space, by letting $\phi(x) = g_j(x) - f_j(x)$ for $x \in A_j$ and $\phi(x) = \theta$ for $x \in K$. It follows from the Tietze Extension Theorem that there exists an extension Φ of ϕ which is a mapping on X into N. We now define a mapping $\psi \in C_n(hK)$ by letting $\psi(x) = R(\Phi(x) + f(x))$. It is easily seen that $\|\psi(x) - f(x)\| < \epsilon$ for all $x \in X$. Moreover, since $\psi(x) = g_j(x)$ for $x \in A_j$, the collection $\psi(A_0), \cdots, \psi(A_n)$ has property D. This proves that $\psi \in G$ and that G is dense in $C_n(h|K)$.

LEMMA 3. *If h is a mapping of a closed subset K of X into I^n, A is a subset of $X - K$ which is closed relative to $X - K$, and $\dim(A) = m \geq n$,*

then there is a residual set (complement of a first category set) $E \subset C_n(h \mid K)$ such that each $f \in E$ is of type $m - n$ on A.

PROOF. We let n be a fixed positive integer and give a proof by induction on $m - n$.

Suppose $m - n = 0$. We consider systems $(V_0, \cdots, V_n)$ of open sets such that each $\overline{V}_j$ is a subset of $X - K$, $V_j \supset \overline{V}_{j+1}$, and the dimension of $A \cap (\text{boundary } V_j)$ is less than n for $j = 0, \cdots, n$. There exists a sequence $(V_{1,0}, V_{1,1}, \cdots, V_{1,n}), (V_{2,0}, V_{2,1}, \cdots, V_{2,n}), \cdots$ of such systems such that if $x \in A$ and U is a neighborhood of x, then for some positive integer p, $x \in V_{p,n}$ and $V_{p,0} \subset U$. We let $A_{p,j} = A \cap (\text{boundary } V_{p,j})$. Each set $A_{p,j}$ is closed in X.

It follows from Lemma 2 that for each positive integer p there exists an open, dense subset G_p of $C_n(h \mid K)$ such that if $f \in G_p$ then $f(A_{p,0}), f(A_{p,1}), \cdots, f(A_{p,n})$ has property D. We let $E = \bigcap_{p=1}^{\infty} G_p$. E is clearly a residual subset of $C_n(h \mid K)$.

Now let $f \in E$, and let $y \in I^n$. We must prove dim $(A \cap f^{-1}(y)) \leqq 0$. Suppose that $x \in A \cap f^{-1}(y)$ and that U is a neighborhood of x. There exists a positive integer p such that $x \in V_{p,n}$ and $V_{p,0} \subset U$. Since $\bigcap_{j=0}^{n} f(A_{p,j}) = \varnothing$, there exists j such that $x \in V_{p,j} \subset U$ and $A_{p,j} \cap f^{-1}(y) = \varnothing$. This proves that $(A \cap f^{-1}(y)) \cap (\text{boundary } V_{p,j}) = \varnothing$ and hence dim $(A \cap f^{-1}(y)) \leqq 0$. Thus f is of type 0 on A.

Let us now assume that the lemma is true for $m - n \leqq k$, and show that this implies the lemma for $m - n = k + 1$. We assume dim $(A) = n + k + 1$. There exists a sequence $W_1, W_2, \cdots$ of open sets such that each $\overline{W}_q$ is a subset of $X - K$, the dimension of $A \cap (\text{boundary } W_q)$ is less than or equal to $n + k$ for each q, and if x is a point of A and U is a neighborhood of x then $x \in W_q \subset U$ for some q. By the inductive hypothesis, for each positive integer q there exists a residual subset E_q of $C_n(h \mid K)$ such that if $f \in E_q$ then f is of type k on $A \cap (\text{boundary } W_q)$. We define $E = \bigcap_{q=1}^{\infty} E_q$. E is a residual set, and it follows that for each $f \in E$, each $y \in I^n$ and each $x \in A \cap f^{-1}(y)$ there exist arbitrarily small neighborhoods of x whose boundaries relative to $A \cap f^{-1}(y)$ have dimension less than $k + 1$. Thus dim $(A \cap f^{-1}(y)) \leqq k + 1$ and f is of type $k + 1$ on A.

THEOREM 1. *If X is m-dimensional, $m \geqq n$, and h is a mapping of a closed subset K of X into I^n, then there is a residual set $E \subset C_n(h \mid K)$ such that f is of type $m - n$ on $X - K$ for each $f \in E$.*

PROOF. Use Lemma 3, letting $A = X - K$.

COROLLARY 1. *If X is m-dimensional, $m \geqq n$, and h is a mapping*

of a closed subset K of X into I^n, then there exists a continuous extension f of h such that f is of type $m-n$ on $X-K$.

PROOF. Use Theorem 1, and the fact that E is nonempty since $C_n(h|K)$ is a complete metric space.

COROLLARY 2. *If X is m-dimensional, $m \geq n$, and h is a mapping of a closed subset K of X into I^n such that h is of type $m-n$ on K, then h can be extended to a mapping f on X into I^n such that f is of type $m-n$ on X.*

PROOF. Let f be the mapping whose existence is asserted in Corollary 1. Then f is of type $m-n$ on each of the sets K and $X-K$. Thus, for each $y \in I^n$, the sets $K \cap f^{-1}(y)$ and $(X-K) \cap f^{-1}(y)$ have dimension equal to or less than $m-n$. Since $K \cap f^{-1}(y)$ is a closed set and $(X-K) \cap f^{-1}(y)$ is an F_σ set, the sum theorem implies that dim $(f^{-1}(y)) \leq m-n$. Thus f is of type $m-n$ on X.

COROLLARY 3. *If X has dimension n and h is a light mapping of a closed subset K of X into I^n, then h can be extended to a light mapping f of X into I^n.*

3. A characterization theorem. In this section we generalize the result obtained in [1].

THEOREM 2. *If X is compact and $m \geq n$, then dim $(X) \leq m$ if and only if there exists a mapping of X into I^n of type $m-n$.*

PROOF. If a mapping of type $m-n$ exists, then it follows from a theorem in [3, p. 91, Theorem VI 7] that dim $(X) \leq m$.

If dim $(X) \leq m$, then we may take $K = \phi$ in Theorem 1 and obtain a residual set $E \subset C_n$ such that each $f \in E$ is of type $m-n$ on X. Since E is nonempty, this proves the existence of mappings of type $m-n$.

COROLLARY 4. *Let X be a compact metric space. Then dim $(X) \leq n$ if and only if there exists a light mapping of X into I^n.*

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