ON A CONVERSE OF THE HÖLDER INEQUALITY

ERNEST B. LEACH

Let X be a measure space with measure μ. We consider the associated dual metric spaces L^p and L^q, where $1<p<\infty$, and $1/p+1/q=1$. By the Hölder inequality, the product of an L^p function with an L^q function is integrable. The following converse is not always true: if f is a measurable function whose product with every L^q function is integrable, then f belongs to L^p. The purpose of this note is to show that this converse is true for a given space X if and only if every measurable subset E of X, of measure $+\infty$, contains a measurable subset F, such that $0<\mu(F)<+\infty$.

It is useful to know that this condition implies the apparently stronger one: if $\mu(E)=+\infty$, then E contains a measurable subset F such that $\mu(F)=+\infty$, and F is a countable union of sets of finite measure. To prove this, let $M=\sup\{\mu(F)\mid F\subset E \;\&\; \mu(F)<\infty\}$ (from here on we assume tacitly that all sets and functions referred to are measurable). Let $\{F_n\}$ be a sequence of subsets of E, such that $\mu(F_n)<\infty$ and $\lim_n \mu(F_n)=M$; and let $F=\bigcup_n F_n$. Then $\mu(F)\geqq M$, because $\mu(F)\geqq\mu(F_n)$ for all n. The stronger condition will be implied if we show that $\mu(F)=+\infty$. If not, then $\mu(E-F)=+\infty$, since $\mu(E)=+\infty$. By the assumed weaker condition on X, $E-F$ contains a subset F' of finite but positive measure. Then, if $\mu(F)<+\infty$, we have $M<\mu(F\cup F')<+\infty$, contrary to the choice of M.

To prove necessity of the condition, we note that if X contains a subset E, of measure $+\infty$, whose subsets have measure 0 or $+\infty$, then the characteristic function of E is evidently not L^p. But if g is any L^q function, the set $\{x\mid g(x)\neq 0\}$ is a countable union of sets of finite measure, and so intersects E in a set of measure 0. So the product of g with the characteristic function of E is 0 almost everywhere and thus is integrable, giving a denial of the converse of the Hölder inequality.

Before proving sufficiency, we shall show that if f is any measurable function whose product with every L^q function is integrable, and if $\{x\mid f(x)\}\neq 0$ is a countable union of sets of finite measure, then $f\in L^p$, independent of any condition on X. In this case there is a sequence, $\{f_n\}$, of functions each of which is bounded and vanishes except on a set of finite measure, and such that the sequence $\{|f_n|^p\}$ converges monotonely to $|f|^p$. Each of the functions f_n determines a

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bounded linear functional on the space L^q, whose value for any g in L^q is given by $(f_n, g) = \int f_n g d\mu$, and whose norm is the L^p norm of the function f_n, namely $(\int |f_n|^p d\mu)^{1/p}$. Also for any g fixed in L^q, the sequence of values of the functionals is bounded, since $|(f_n, g)| \leqq \int |fg| d\mu < \infty$, because $|f_n| \leqq |f|$ and the product fg is integrable by hypothesis. Therefore the Banach-Steinhaus theorem applies, and the sequence of norms of the functionals is bounded. Then $\int |f_n|^p d\mu \leqq M$, and since $|f_n|^p$ converges monotonely to $|f|^p$, $\int |f|^p d\mu < \infty$. This proves the assertion.[1]

We can now prove sufficiency of the condition on X. Assuming this condition, let f be any function whose product with every L^q function is integrable. According to the preceding paragraph, we need only show that $\{x | f(x) \neq 0\}$ is a countable union of sets of finite measure. Let $E_n = \{x | |f(x)| \geqq 1/n\}$, $n = 1, 2, \cdots$. We shall show that $\mu(E_n) < \infty$ for all n, which will prove the assertion. If not, then $\mu(E_m) = +\infty$ for some integer m, and by the "stronger" version of the condition on X, E_m contains a subset F, such that $\mu(F) = +\infty$, and F is a countable union of sets of finite measure. The product of f and the characteristic function of F satisfies the condition of the preceding paragraph and so belongs to L^p. But this is a contradiction because $|f| \geqq 1/m$ on F, and $\mu(F) = +\infty$. This proves our theorem.

In closing we give a simple example of a measure space that satisfies our condition but which is not a countable union of sets of finite measure. If X is any noncountable set, we assign to every finite subset of X a measure equal to the number of points in that set; to all other sets we assign the measure $+\infty$.

CASE INSTITUTE OF TECHNOLOGY

[1] This proof was given by Professor Walter Rudin in a course at Massachusetts Institute of Technology in 1951. The author is also indebted to the referee for observing that a proof for the case when X is of finite measure may be found on p. 336 of Dunford, *Uniformity in linear spaces*, Trans. Amer. Math. Soc. vol. 44 (1938) pp. 305–356.