DIFFERENTIABLE FUNCTIONS, FORMAL POWER SERIES, AND MOMENTS

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Every maximal ideal in the algebra $\mathfrak{D}^\infty$ of infinitely-differentiable functions on an n-dimensional compact manifold $\mathfrak{X}_n$ is of the form $M(x) = \{f \in \mathfrak{D}^\infty : f(x) = 0\}$ for some fixed $x \in \mathfrak{X}_n$. The smallest primary ideal in $M(x)$, i.e. the smallest ideal contained in no maximal ideal other than $M(x)$, is $J(x) = \{f \in \mathfrak{D}^\infty : f \equiv 0$ on some neighborhood V_f of $x\}$. These two facts more or less exhaust our purely algebraic knowledge of the ideals in $\mathfrak{D}^\infty$. But $\mathfrak{D}^\infty$ has also a natural complete metric vector space topology under which a series $\sum f_k$ is said to converge if for each differential operator D the derived series $\sum Df_k$ converges uniformly. The closure of $J(x)$ in this topology is the smallest closed primary ideal in $M(x)$, and every closed ideal is the intersection of closed primary ideals. See Whitney [**4**], Schwartz [**3**]. $\bar{J}(x)$ can be described concretely as the f that vanish together with all derivatives at x. (And most sophomore calculus students are convinced that $\bar{J}(0)$ contains precisely two elements: e^{-1/x^2} and 0.)

If $h_1, \cdots, h_n \in M(x)$ are local coordinate functions on $\mathfrak{X}_n$, then their images $X_1, \cdots, X_n$ generate the quotient algebra $\mathfrak{D}^\infty/\bar{J}(x)$ and the mapping $\mathfrak{D}^\infty \to \mathfrak{D}^\infty/\bar{J}(x)$ sends every f into its formal taylor series expansion in these coordinates. The following proposition shows that $\mathfrak{D}^\infty/\bar{J}(x)$ is actually the algebra of *all* formal power series in $X_1, \cdots, X_n$. The one-dimensional case was proved by E. Borel [**1**] half a century ago, and recently re-proved by Rosenthal [**2**]. Although the n-dimensional case is well-known to persons interested in the behavior of derivatives, it seems never to have been written down publicly. And the present proof has some merit even for $n=1$.

PROPOSITION 1. *Let $\lambda_{p_1,\cdots p_n}$ be an arbitrary family of complex numbers, indexed by n-tuples $(p_1, \cdots, p_n)$ of non-negative integers. Then there exists some infinitely-differentiable function f of n real variables that has as its derivatives at the origin exactly these complex numbers.*

$$\left(\frac{\partial}{\partial \xi_1}\right)^{p_1} \cdots \left(\frac{\partial}{\partial \xi_n}\right)^{p_n} f(0, \cdots, 0) = \lambda_{p_1 \cdots p_n}.$$

It will be convenient to make n-dimensional statements in a one-dimensional notation. For any n-tuple $P = (p_1, \cdots, p_n)$ define

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$|P| = p_1 + \cdots + p_n$. For any point $x = (\xi_1, \cdots, \xi_n)$ in real n-space and any $f \in \mathfrak{D}^\infty$ write $f^{(P)}(x)$ instead of

$$\left(\frac{\partial}{\partial \xi_1}\right)^{p_1} \cdots \left(\frac{\partial}{\partial \xi_n}\right)^{p_n} f(\xi_1, \cdots, \xi_n).$$

We are looking for an f that satisfies all the conditions

$$f^{(P)}(0) = \lambda_P.$$

Since we are concerned only with what happens near the origin, every f that appears in our calculations will be assumed to vanish for $|x| \geqq 1$. Write $\|f\|$ for $\sup_x |f(x)|$.

PROOF OF PROPOSITION 1. We shall construct by induction a sequence f_k of functions satisfying

$$f_0^{(P)}(0) + \cdots + f_k^{(P)}(0) = \lambda_P, \qquad \text{for } |P| = k, \tag{1}$$

$$f_k^{(P)}(0) = 0, \qquad \text{for } |P| < k, \tag{2}$$

$$\|f_k^{(P)}\| < 2^{-k}, \qquad \text{for } |P| < k. \tag{3}$$

We can then define $f = \sum f_k$. The series will converge in $\mathfrak{D}^\infty$ by condition (3), and its derivatives can be evaluated term by term, giving $f^{(P)}(0) = f_0^{(P)}(0) + \cdots + f_{|P|}^{(P)}(0) + \sum_{k>|P|} f_k^{(P)}(0) = \lambda_P + 0$, Q.E.D.

To construct f_0 is trivial. And given $f_0, \cdots, f_{k-1}$ then an f_k satisfying (1) and (2) can be manufactured by multiplying a suitable polynomial times a function $\equiv 1$ near the origin and $\equiv 0$ for $|x| \geqq 1$. If now f_k does not satisfy (3) replace it by $\rho^{-k} f_k(\rho x) = \rho^{-k} f_k(\rho\xi_1, \cdots, \rho\xi_n)$ with some large positive ρ. The new f_k still satisfies (1) and (2) and has $\|f_k^{(P)}\|$ as small as we like for $|P| < k$.

With the help of various series and integral transforms, Proposition 1 can be used to solve moment problems. Proposition 2 below is offered as a sample. It is exactly the fourier transform of Proposition 1, except that the family λ_P must be modified to compensate for multiplication by $i^{|P|}$. Recall that the transform of a compact-supported function is analytic, and that for every $m > 0$ the transform of an infinitely-differentiable function vanishes faster at infinity than $|x|^{-m}$.

PROPOSITION 2. *Given an arbitrary family $\lambda_{p_1 \cdots p_n}$ of complex numbers indexed by n-tuples of non-negative integers, there exists an analytic function f whose $(p_1, \cdots, p_n)$th moment is $\lambda_{p_1 \cdots p_n}$ for each $(p_1, \cdots, p_n)$.*

$$\int \cdots \int f(\xi_1, \cdots, \xi_n)\xi_1^{p_1} \cdots \xi_n^{p_n} d\xi_1 \cdots d\xi_n = \lambda_{p_1 \cdots p_n}.$$

All integrals are absolutely convergent.

References

1. E. Borel, *Sur quelques points de la théorie des fonctions*, Ann. École Norm. (3) vol. 12 (1895) p. 44.

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3. L. Schwartz, *Analyse et synthèse harmoniques dans les espaces de distributions*, Canadian Journal of Mathematics vol. 3 (1951) p. 506.

4. H. Whitney, *On ideals of differentiable functions*, Amer. J. Math. vol. 70 (1948) pp. 634–658.

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